Projection algorithms and convergence theory

E.A. Nurminski

Institute for Automation and Control Processes
Vladivostok, Russia
nurmi@dvo.ru

Universita della Calabria
December 2008

Convex optimization problem

$$\min f(x) = -f^\star(0) = \sup_x \{0x - f(x)\}$$

Algorithmic idea

Reduce computation of $f^\star(0)$ to projections on approximation of epi $f^\star$.

If it were possible to project on epi $f^\star$ itself then we would have superlinear convergence !

Least distance problem:

$$\min_{\substack{x = \sum_{i=1}^{m} \lambda_i \hat{x}^i \\ (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Delta_m}} \|x\| = \|x^\star\| = \| \sum_{i=1}^{m} \lambda_i^\star \hat{x}^i \| \qquad (1)$$

Wolfe algorithm:

1. For *certain* $I \subset \{1, 2, \dots, m\}$ solve (**??**) without nonnegitivity constraint. If some $\lambda_i^\star$ is negative, drop it from I according to some rule and resolve.
2. For $x^\star = \sum_{i \in I} \lambda_i^\star \hat{x}^i$ find k_I such that $\hat{x}^{k_I} x^\star = \min \hat{x}^i x^\star$ and add it to corral: $I \to I \cup k_I$.

Type of convergence:

$$\|x^\star\| \leq Cq^k, k = 0, 1, \ldots$$

where k is a number of iterations.

If $0 \notin \mathrm{co}\,\{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^m\}$ convergence is finite and "better then linear".

Precise upper bound for q is unknown.

Least norm solution for a system of inequalities:

$$\min_{Ax \leq b} \tfrac{1}{2}\|x\|^2 = \|x^\star\|$$

A — $m \times n$ matrix, etc.
Appyling exact penalty: there exists $\Gamma > 0$ such that for all $\gamma \geq \Gamma$

$$\min_{Ax \leq b} \tfrac{1}{2}\|x\|^2 = \min\{\frac{1}{2}\|x\|^2 + \gamma|Ax - b|_\infty^+\} \tag{2}$$

where $|Ax - b|_\infty^+ = \max\{0, \max_{i=1,2,\ldots,n}(Ax - b)_i\}$.

Denote $\bar{x} = (x, x_{n+1})$ and $\bar{A} = \|A|b\|$.
Then

$$Ax \leq b \leftrightarrow \bar{A}\bar{x} \leq 0, \bar{x}_{n+1} = 1$$

Moreover

$$|Ax - b|_\infty^+ = |\bar{A}\bar{x}|_\infty^+ = \max_{\lambda \in \Delta_m} \operatorname{co}\{0, (\bar{A}\bar{x})_i\}, i = 1, 2, \ldots, n\}.$$

Therefore in terms of support function:

$$|Ax - b|_\infty^+ = (\operatorname{co}\{0, (\bar{A})_i\}, i = 1, 2, \ldots, n\})_{\bar{x}}$$

with $\bar{x} = (x, 1)$.

Rewriting penalty term:

$$\min_{Ax \leq b} \tfrac{1}{2}\|x\|^2 = \min\{\tfrac{1}{2}\|\bar{x}\|^2 + \gamma(\mathrm{co}\,\{0, (\bar{A})_i\}, i = 1, 2, \ldots, n\})_{\bar{x}}\} =$$
$$\min\{\tfrac{1}{2}\|\bar{x}\|^2 + (\gamma\mathrm{co}\,\{0, (\bar{A})_i\}, i = 1, 2, \ldots, n\})_{\bar{x}}\} =$$
$$\min\{\tfrac{1}{2}\|\bar{x}\|^2 + (D_\gamma)_{\bar{x}}\}$$

where $D_\gamma = \gamma\mathrm{co}\,\{0, (\bar{A})_i\}, i = 1, 2, \ldots, n\}$ and again $\bar{x} = (x, 1)$.

Using Lagrange relaxation on $\bar{x}_{n+1} = 1 = \bar{x} e^{n+1}$ obtain

$$\min_{Ax \leq b} \tfrac{1}{2}\|x\|^2 = \max_u \min_{\bar{x}} \{ \tfrac{1}{2}\|\bar{x}\|^2 + (D_\gamma)_{\bar{x}} + u(\bar{x} e^{n+1} - 1) \} - \tfrac{1}{2} =$$
$$\max_u \{ -u + \min_{\bar{x}} \{ \tfrac{1}{2}\|\bar{x}\|^2 + (D_\gamma + u e^{n+1})_{\bar{x}} \} \} - \tfrac{1}{2}$$

The essential part of above is

$$\phi(u) = -\min_{\bar{x}} \{ \frac{1}{2}\|\bar{x}\|^2 + (D_\gamma + u e^{n+1})_{\bar{x}} \} \} = \min_{\bar{x} \in D_\gamma + u e^{n+1}} \tfrac{1}{2}\|\bar{x}\|^2$$

where $D_\gamma = \gamma \mathrm{co}\,\{0, (\bar{A})_i\}, i = 1, 2, \ldots, n\}$ with γ arbitrary large.

It can be shown that for "$\gamma = \infty$"

$$\phi(u) = \alpha u^2, \alpha = \phi(1) > 0$$

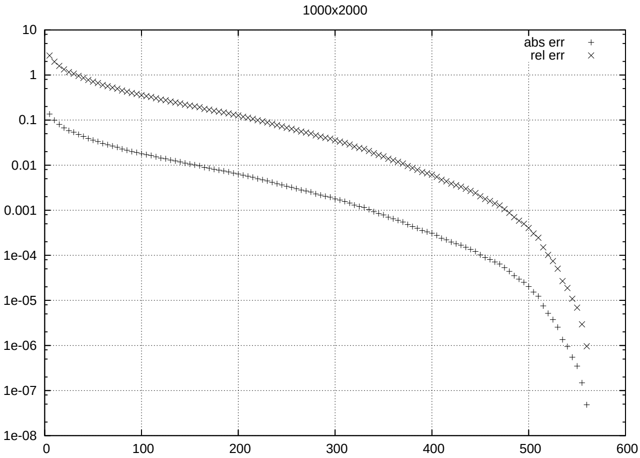
and hence

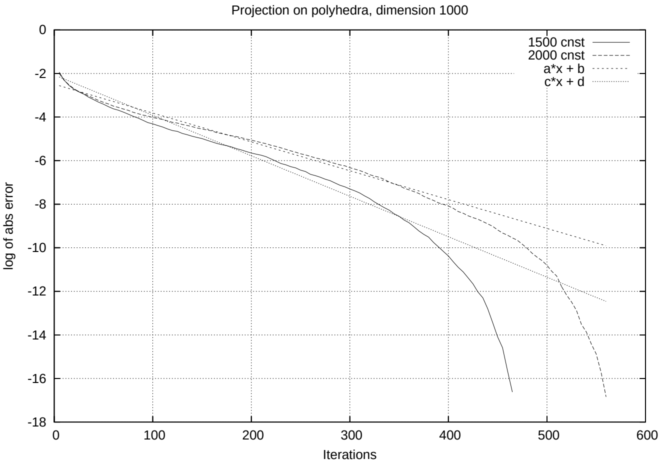
$$\begin{array}{c} \min \frac{1}{2}\|x\|^2 \\ Ax \leq b \end{array} = -\frac{1}{4\phi(1)}.$$

that is it is sufficent to solve the polytope-like problem

$$\begin{array}{c} \min \frac{1}{2}\|\bar{x}\|^2 \\ \bar{x} \in \mathrm{Co}\{\bar{A}_i, i = 1, 2, \ldots, m\} + e^{n+1} \end{array}$$

with with only m rays and $n+1$ variables.


1000x2000
abs err
rel err
10
1
0.1
0.01
0.001
1e-04
1e-05
1e-06
1e-07
1e-08
0
100
200
300
400
500
600


Projection on polyhedra, dimension 1000
1500 cnst
2000 cnst
a*x + b
c*x + d
log of abs error
Iterations
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
0
100
200
300
400
500
600