Projection and NDO

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Problem formulation

Orthogonal projection problem:

$$\min \|x\|^2$$
$$x \in X$$

X — polyhedra, polytope, polyhedral cone, and so on, obtained from abovementioned.

Motivation: optimization methods, automatic classification, computer tomography, etc.

Problems: high dimensionality (10^7 — typical), request for high accuracy and relatively small solution times, ill-conditioned and semi-definite problems, many different data representations.

Connection with NDO

Convex analysis:

$$\min_{x \in X} \tfrac{1}{2}\|x\|^2 \quad = -\min_{x}\{\frac{1}{2}\|x\|^2 + \sigma_X(x)\}$$

where $\sigma_X(x)$ — support function of X:

$$\sigma_X(x) = \sup_{z \in X} xz.$$

Easy to remember: take X to be a singleton $\{a\}$. Then

$$\frac{1}{2}\|a\|^2 = -\min\{\frac{1}{2}\|x\|^2 + xa\} = -\{\frac{1}{2}\|a\|^2 - \|a\|^2\} = \frac{1}{2}\|a\|^2$$

Examples

P — polytope, Q — polyhedra, K — polyhedral cone, $A, B, \ldots$ — convex sets.

- $X = A - B$ — shortest distance between sets (Jh. Von Neumann);
- $X = P \cap Q$ — nonsmooth optimization (additional cuts in bundle-type methods);
- $X = P + K$ — constrained nonsmooth optimization (Dubovitsky-Milutin theorem);
- $X = \mathrm{co}\{A, B, \ldots\}$ — internal decomposition;
- $X = A \cap B \cap \ldots$ — external decomposition;
- $X = \{x : Ax + By \leq b\}$ — implicit representation.

Basic model — polytope

Defined as convex set, determined by its vertices:

$$X = \mathrm{co}\{\hat{x}^0, \hat{x}^1, \dots, \hat{x}^M\}$$

Least norm problem:

$$\min_{x \in X} \|x\|^2 = \min_{x = \hat{X}\lambda,\ \lambda \in \Delta_M} \|x\|^2$$

Δ_M — standard M-dimensional simplex: $\Delta_M = \{\lambda_i \geq 0, \sum_{i=0}^{M} \lambda_i = 1\}$, $\hat{X}$ — $n \times (M+1)$ matrix $\|\hat{x}^0 \hat{x}^1 \dots \hat{x}^M\|$.

Troubles: semidefinite objective, non-unique λ.

Simplification ?

$$\begin{array}{ccccc} \min \|x\|^2 & = & \min \lambda' \hat{X}' \hat{X} \lambda & = & \min \lambda' G \lambda \\ x = \hat{X}\lambda, & & \lambda \in \Delta_{M-1} & & \lambda \in \Delta_{M-1} \\ \lambda \in \Delta_{M-1} & & & & \end{array}$$

As a result:

+ Fewer variables (no x);
– Degenerosity of G when $M > n$, , higher density of G in comparison with $\hat{X}$.

Reviews of basic algorithms

1. Еремин И.И., Васин В.В. Фейеровские операторы М.:Наука, 2003
2. H.H. Bauschke, J.M. Borwein On projection algorithms for solving convex feasibility problems, SIAM Review 38(3), pp. 367-426, 1996
3. Y. Censor, W. Chen, P.L. Combettes, R. Davidi and G.T. Herman, On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints, Computational Optimization and Applications,

Suitable affine subspaces

Definition

The set $S_I = \mathrm{co}\{x^i, i \in I\}$, determined by index set $I \subset \mathcal{N}$ is called a suitable sub-simplex if

$$\min_{z \in \mathrm{aff}\{S_I\}} \|z\|^2 = \min_{z \in S_I} \|z\|^2.$$

Any singleton set I will determine a suitable (0-dimensional) suitable sub-simplex.

Definition

The affine hull of a suitable sub-simplex is called a suitable affine subspace.

Affine subspace method

Balder v. Hohenbalken (1975), Ph. Wolfe (1976) and possibly earlier.

Main computational loop

I_k — current basis, $_k$ — affine subspace, generated by I_k.

1. **Project on subspace H_k.** Solve problem

$$\min_{z \in H_k} \|z\|^2 = \|z^k\|^2.$$

2. **Optimality check**
3. Take any $i_k \in \mathcal{N} \setminus I_k$ such that $x^{i_k} z^k < \|z^k\|^2$. Add x^{i_k} in the current basis I_k.
4. **Internal loop.** Basis correction.
5. **End of main loop.**

Internal loop.

J_s — current basis, G_s — affine subspace, generated by J_s. At the beginning $J_0 = I_k$.

1. **Projection on modified basis.**

$$\min_{y \in G_s} \|y\|^2 = \|y^s\|^2.$$

2. **Feasibility check.**
3. **Basis correction.** (Performed in the case when $y^s \notin T_s$.) Let

$$u(\mu) = \mu y^s + (1 - \mu) w^s$$

   and determine the maximal μ_s such that $u(\mu_s) \in T_s$. The point $u(\mu_s)$ belongs to relative interior of a certain minimal face of sub-simplex T_s, and define a new basis J_{s+1} and new G_{s+1}.

The end of internal loop.

Numerical experiment

Test dataset:

$$\xi_i = \begin{cases} \sigma(\zeta_i - 0.5) & i = 1, 2, \ldots, n-1 \\ \sigma^{-1}\zeta_n + \delta & i = n \end{cases},$$

where σ — scaling factor, δ — the shift in the direction of n-th coordinate, $\zeta_i, i = 1, 2, \ldots, n$ — random i.u.d. on $[0, 1]$

Simplest algorithm (Demyanov, Malozemov 1974): problem size 100x100, more then 1 million iterations, accuracy of the order of 10^{-2}.

Projection on 2000x1999 dataset

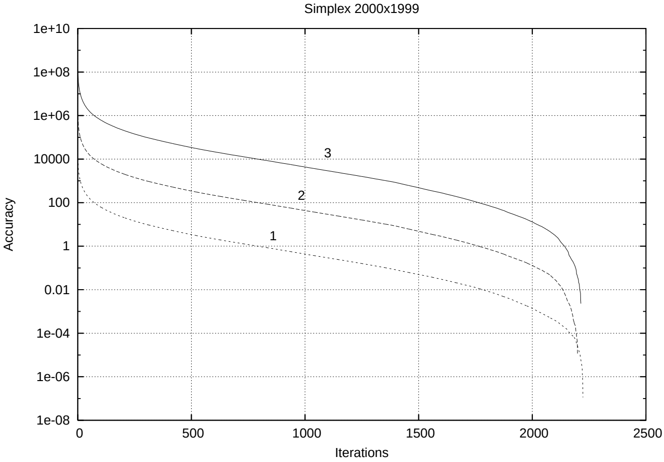

Рис.: Polytope 1999x2000. Convergence in norm. Graphs: 1: $\sigma^2 = 10$, 2: $\sigma^2 = 1000$, 3: $\sigma^2 = 10000$, shift parameter $\delta = 0.001$.

Projection on 2000x1999 dataset

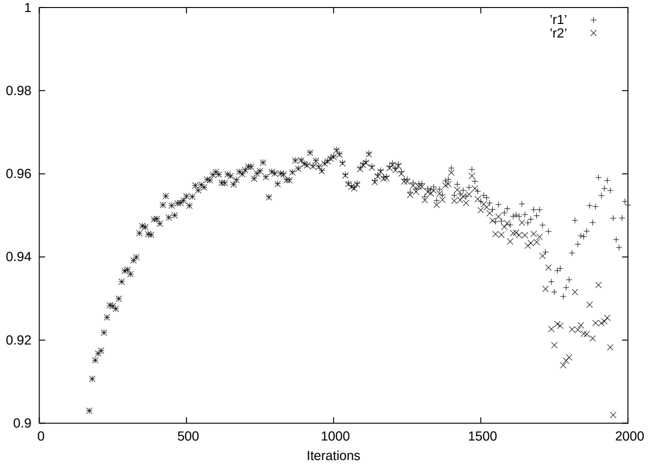

Рис.: Polytope 1999x2000. Convergence in norm. Linear rate convergence multiplier.

Projection on polyhedra

Convex polyhedra, determined by the linear inequalities:

$$X = \{x : Ax \leq b.\}$$

Least norm element:

$$\min_{x \in X} \tfrac{1}{2}\|x\|^2 = \min\{\tfrac{1}{2}\|x\|^2 + C|Ax - b|_1^+\} = \min\{\tfrac{1}{2}\|\bar{x}\|^2 + C \max_{p \in P} \bar{x}p\}$$

where $\bar{x} = (x, x_{n+1}), \bar{A} = \|A\vdots - b\|$, $\bar{x}_{n+1} = 1$, and $P = \mathrm{co}\{\{\bar{A}'\lambda, \lambda \in \Delta\}, 0\}$ Thanks to convex analysis

$$\min\{\frac{1}{2}\|\bar{x}\|^2 + C \max_{p \in P} \bar{x}p\} = -\min_{\bar{x} \in CP} \tfrac{1}{2}\|\bar{x}\|^2,$$

If C is large enough solution does not depend on C so CP can be replaced by the cone $K_{A,b}$ generated by rows of the matrix $\bar{A}$.

Internal decomposition

$$X = \{x = \sum_{i=1}^{M} \lambda_i X^i, \lambda \in \Delta_{M-1}\} = \mathrm{co}\{X^i, i = 1, 2, \ldots, M\}$$

Decomposition algorithm (z — approximate solution):

- For any $i = 1, 2 \ldots, M$ find least distance elements in $\bar{X}_i = \mathrm{co}\,\{z, X_i\}$. Obtain $\bar{x}^i, i = 1, 2, \ldots, M$.
- Form the aggregate set $Z = \mathrm{co}\{\bar{x}^i, i = 1, 2, \ldots, M.\}$. and find least norm element z in Z and repeat.

Statement: $z \to z^\star$, the solution of the complete problem $\min \|x\|, x \in X$.

For a polytope $X = \mathrm{co}\{x^j, j = 1, 2, \ldots, N\}$ internal decomposition can be defined with the help of sets

$$X_i = \mathrm{co}\{x^j, j \in C_i, i = 1, 2, \ldots, M\};$$

where $C_i, i = 1, 2, \ldots, M$ — certain coverage of $1, 2, \ldots, N$.

Convergence can be greatly improved if sets C_i can be modified during the process.

External decomposition, CFP

Find a feasible point belonging to the set

$$X = \Cap_{i=1}^{M} X^i, i = 1, 2, \dots$$

$X \subset X_i, i = 1, 2, \dots, M$ — supersets, generating X.

Sequential projection method:

- For $i = 1, 2, \dots, M$ solve problems $\min_{x \in X_i} \|x^{i-1} - x\| = \|x^{i-1} - x^i\|$. Set $x^0 = x^M$ and repeat.

Parallel projection method:

- For $i = 1, 2, \dots, M$ solve $\min_{x \in X_i} \|z - x\| = \|z - x^i\|$. Let new approximate solution to be $z = \sum_{i=1}^{M} w_i x^i$, where $w \in \Delta_{M-1}$ repeat.