Row-Oriented Decomposition in Large-Scale Linear Optimization [1]

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Outline

- The problem origin and challenges
- New ideas and prelinitary tests
- Nasty details and tricks of the trade
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Problem Origin

Once upon a time there was a transportation company which operated the fleet of of more than 50000 railroad cars over the territory more tnan 17 mln km^2. To make it efficiently the company finaly decided to advance its decision support system with the mathematical core aimed at profit maximization ...

Challenges

The key part of this core are more or less traditional linear optimization problems but their shear size makes them unsolvable in practice by off-the-shelf solvers. The plethora of models in production use looks like follow

Planning horisont (days)	5	10	15
Variables	29108202	53404781	77679826
Constraints	210580	385322	559957
Nonzeros	58570131	107412709	152218630

Model: Balance Constraints (Part 1)

$1, 2, \ldots, T$ – days under consideration;
S – stations, K – car types, E – empty car routes, Q – departure schedule;
x, y, z – variables; q, τ, θ, σ – input parameters.

Daily Dynamics of the Railway Car Flows:

$$y_{sk}^{t} - y_{sk}^{t-1} - \sum_{\substack{r \in E: \\ r = (\cdot \to s)}} x_{rk}^{t-\theta_r} + \sum_{\substack{r \in E: \\ r = (s \to \cdot)}} x_{rk}^{t} -$$

$$- \sum_{\substack{(\varsigma - \tau_{r(o)}, o) \in Q: \\ r(o) = (\cdot \to s)}} \sum_{\substack{i = -\sigma_o: \\ \varsigma + i = t}}^{\sigma_o} z_{oki}^{\varsigma - \tau_r} + \sum_{\substack{(\varsigma, o) \in Q: \\ r(o) = (s \to \cdot)}} \sum_{\substack{i = -\sigma_o: \\ \varsigma + i = t}}^{\sigma_o} z_{oki}^{\varsigma} = q_{sk}^{t}$$

$s \in S,\ k \in K,\ t \in 1, 2, \ldots, T.$
$|S| = 1045,\ |K| = 32,\ T = 10$

Model: Balance Constraints (Part 2)

$\bar{x}, \bar{y}, \bar{z}$ – variables; $\bar{q}, \tau, \theta, \bar{O}$ – input parameters.

Daily Dynamics of the Railway Car Flows:

$$\bar{y}_{sk} - y_{sk}^{T} - \sum_{\substack{r \in E \\ r = (\cdot \to s)}} \sum_{\substack{t=1 \\ t+\theta_r > T}}^{T} x_{rk}^{t} - \sum_{\substack{r \in E \\ r = (\cdot \to s)}} \bar{x}_{rk} + \sum_{\substack{r \in E \\ r = (s \to \cdot)}} \bar{x}_{rk} -$$

$$- \sum_{\substack{(t-\tau_{r(o)}, o) \in Q : \\ r(o) = (\cdot \to s)}} \sum_{\substack{i = -\sigma_o : \\ t+i > T}}^{\sigma_o} z_{oki}^{t-\tau_{r(o)}} - \sum_{\substack{o \in \bar{O} \\ r(o) = (\cdot \to s)}} \bar{z}_{ok} +$$

$$+ \sum_{\substack{o \in \bar{O} \\ r(o) = (s \to \cdot)}} \bar{z}_{ok} = \bar{q}_{sk}, \quad s \in S, \quad k \in K.$$

Model:Objective Function

$$F(x, y, z, \dots) = F_1(z) - F_2(x) - F_3(y) + \dots.$$

- Total Revenue from Loaded Cars

$$F_1(z) = \sum_{(t,o)\in Q} \sum_{k\in K} \sum_{i=-\sigma_o}^{\sigma_o} (p_o - \xi l_{r(o)}) z^t_{oki};$$

- Total Cost from Empty Cars

$$F_2(x) = \sum_{t=1}^{T} \sum_{r\in E} \sum_{k\in K} (c_r + \xi l_r) x^t_{rk};$$

- Total Cost from Idle Cars

$$F_3(y) = \sum_{t=1}^{T} \sum_{s\in S} \sum_{k\in K} \lambda_s y^t_{sk}.$$

New ideas

Projection algorithm:

$$\min_{x \in P} \; cx \;\leftrightarrow\; \min_{x \in P} \; \|x - x^c\|^2, x^c = x^0 - \tau c.$$

Via exact penalty function:

$$\min_{x \in P} \; \|x\|^2 \;\leftrightarrow\; \min_{x} \; \{\|x\|^2 + (K)_x\} \;\leftrightarrow\; \min_{x \in K} \; \{\|x - a\|^2\}$$

K — the polyhedral cone (dep on $c, P, a = (0, 0, \dots, 0, 1)$).

Very brief history and news:

- Nurminski, E.A.: Single-projection procedure for linear optimization. Journal of Global Optimization **66**(1), 95-–110 (2016)
- Bui, Hoa T., Ryan Loxton, and Asghar Moeini: A note on the finite convergence of alternating projections. Operations Research Letters 49.3 (2021): 431-438.

Demonstration test

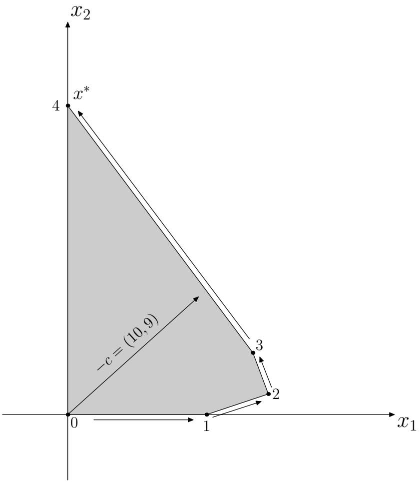
We illustrate solution of LP-problems with a tiny example:

$$\min -10x_1 - 9x_2$$

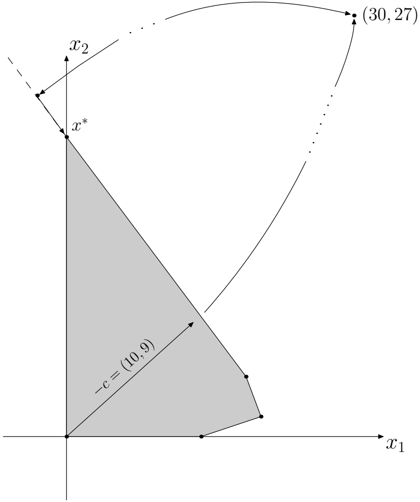
subjected to general constraints

$$\begin{array}{ccccc} 4x_1 & + & 3x_2 & \leq & 5 \\ 8x_1 & + & 3x_2 & \leq & 9 \\ 2x_1 & - & 6x_2 & \leq 3 & \end{array}$$

and nonnegativity constraints $x_1, x_2 \geq 0$.



Simplex

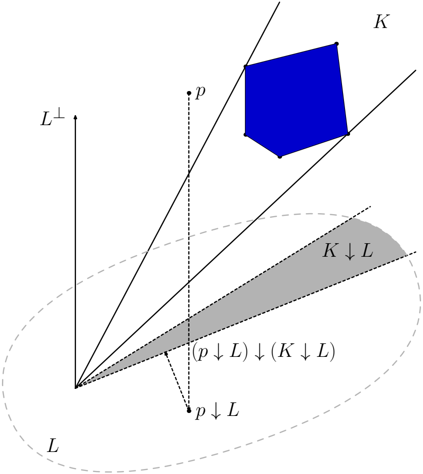


Single-projection procedure, $\theta = 3$

Closer look at optimization problem:

$$\begin{array}{ccc} \min cx & \leftrightarrow & \min \|z - p\|^2 \\ A_E x = b_E & & x \in K_L + L_\perp \\ A_L x \leq b_L & & \end{array}$$

or $z^\star = (p \downarrow L) \downarrow (K \downarrow L)$.



Auxiliary projections

The basic operation:

$$\min_{z \in L} \|z - p\|^2 = \|p \downarrow L - p\|^2$$

where $L = \{z = \bar{A}_E w, w \in E^m\} = A_E E^m$.
Analytical solution

$$p \downarrow L = A_E^T (A_E A_E^T)^{-1} A_E p = A_E^T (RR^T)^{-1} A_E p$$

can be converted into the sequence of matrix-vector operations with sparse matrices

$$u = A_E p \ \rightarrow \ (RR^T) w = u \ \rightarrow \ p \downarrow L = A_E^T w$$

which can be done very quickly and Cholessky factor R can be reused.
Execution times:

- R – 122347×122347, 73253479 nonzeros (0.48%), time 3.517 sec.
- $A_E A_E^T$ — 30006369 nonzeros (0.2%), time 2.056 sec.

Key problem

The final step:

$$\min_{z \in K_L} \quad \|z - p_L\|^2 = \|z_L - p_L\|^2$$

No closed form solution, iterative methods.
QP-formulation (assuming $K_L = \mathrm{Co}(\hat{z}^k, k = 1, 2, \dots)$:

$$\begin{array}{cc} \min & \|z_L - p_L\|^2 \\ z_L = \sum v_k \hat{z}^k & \\ v_k \geq 0 & \end{array}$$

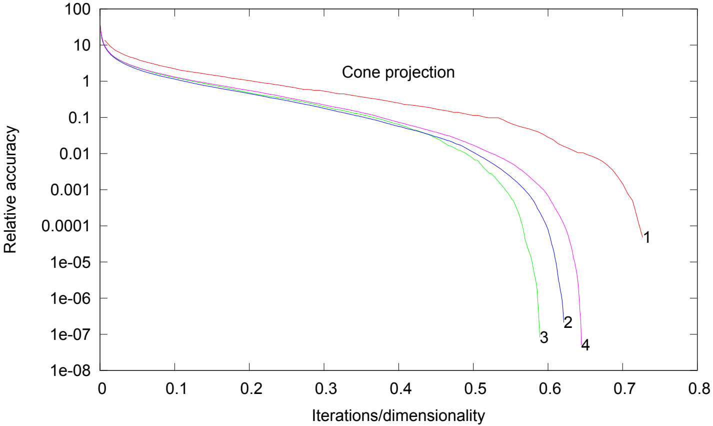
looks hopeless.
More or less workable — conical extension of iterative algorithm of polytope projection (Nurminski E.A., Computational Mathematics and Mathematical Physics, Vol. 45 No. 11, 2005, pp. 1915-1922) which `OCTAVE` implementation lived through tens of refinements and modifications, see `DOI: 10.13140/RG.2.2.12814.08002`.

Projection tests

The parameters of projections problem solved in series of experiments.

nn	Gener	Dim	Iterations	Optimal base	Optimality
1	150	130	109	94	$3.01 \cdot 10^{-13}$
2	550	430	325	303	$4.47 \cdot 10^{-12}$
3	850	630	457	441	$2.84 \cdot 10^{-12}$
4	1500	830	651	647	$2.22 \cdot 10^{-15}$

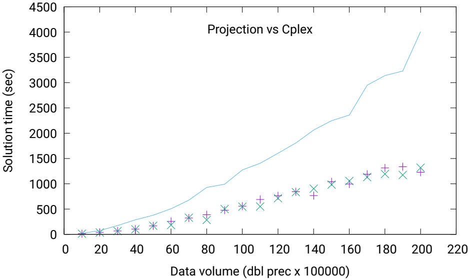
Convergence of the cone projection algorithm



Dense problems

Tests with dense inequalities-only problems

$$\min \quad cx$$
$$A_L \leq b_L$$



Runtime vs problem size

Solid line — CPLEX, crosses – projection algorithm.
Data size measured in 10^5 double precision (8 bytes) numbers.
Rows-columns ratio is 3 : 1.

Conclusions

By any means this is the work in progress. Many technical problems still have to be solved, but it looks like it is a promissing direction for solving giga-scaled linear optimization problems.