Advanced Studies in Contemporary Mathematics, 22 (2012), No. 1, pp. 153-160

Accelerated parallel projection method for solving the shortest distance problem

D.V. Dolgy [1], E.A.Nurminski

Abstract. We consider the problem of finding the vector of minimum length in the simplex of finite dimensional Euclidean space. A final accelerated parallel algorithm for solving this problem is proposed.

Keywords. Simplex, projection, parallel algorithm, convex polyhedron, Euclidean space, affine and convex envelopes.

2000 Mathematics Subject Classification

1. Introduction

The subject of this work is to develop special final procedure for finding the projection on the simplest form of a convex polyhedron - simplex and to construct of a parallel algorithm for solving this problem. Conducted computational experiments demonstrate high computational efficiency of the algorithm.

2. Problem statement

The objects of study are the subsets and vectors of n-dimensional Euclidean space with inner product xy and norm $\|x\|^2 = xx$.

Define for a set $X \subset E^n$ that can be represented as the union of a family of sets $\{X_k, k = 1,2,...,N\}$, i.e. $X = \bigcup_{k=1}^{N} X_k$, affine $aff(X)$ and convex $co(X)$ envelopes as follows:

$$aff(X) = \{x: \ x = \sum_{i=1}^{n+1} \lambda_i x_i, \ \sum_{i=1}^{n+1} \lambda_i = 1, \ x_i \in X, \ i = 1,2,...,n+1\},$$

$$co(X) = \{x: \ x = \sum_{i=1}^{n+1} \lambda_i x_i, \ \sum_{i=1}^{n+1} \lambda_i = 1, \ \lambda_i \ge 0, \ x_i \in X, \ i = 1,2,...,n+1\}.$$

The subject of this paper is the solution of the fundamental problem of finding the distance from the origin to set $Y = co(X)$:

---

[1] d_dol@mail.ru

$$\min_{z \in Y} \|z\|^2 . \tag{1}$$

The only solution to the problem (1) is a vector $z \in Y$ satisfying the following condition (in fact, the variation inequality):

$$z(x - z) \geq 0, \quad \forall x \in Y, \tag{2}$$

and the condition (2) is necessary and sufficient.

The equivalence of (1) and the variation inequality (2) provided the basis for the development of many of projective algorithms for solving variation inequalities [1,4,5].

3. The method of affine subspaces

To solve (1) when $N \leq n+1$ a suitable affine subspaces method has been developed [3]. The method starts with the entry of a suitable base $I_0 \subset \Im = \{1,2,...,N\}$ and completes its work with the construction of the optimal basis $I_* \subset \Im$ and accordingly $Y_* = co(X_i, i \in I_*)$ for which $\min_{z \in Y_*} \|z\|^2 = \min_{z \in Y} \|z\|^2$.

In the course of the algorithm a sequence of bases $I_k, k = 1,2,...,$ and the corresponding sequence of sub-simplexes $Y_k = co(X_i, i \in I_k)$ which guarantees the monotone decrease of the distance at a geometric rate $q\rho_k^2 = \min_{z \in Y_k} \|z\|^2 > \rho_{k+1}^2 = \min_{z \in Y_{k+1}} \|z\|^2$ where $q \in [0,1)$ are constructed.

Iterative transition from the basis I_k to the basis I_{k+1} is in the following two basic steps:

Step 1. Solve the problem $\min_{z \in H_k} \|z\|^2 = \|z^k\|^2$, where H_k is affine subspace: $H_k = aff(x_i, i \in I_k)$. If z^k satisfies (2) then it is a solution of (1) and the algorithm terminates. Otherwise there is a vector x^{i_k} for which $x^{i_k} z^k < \|z^k\|^2$ and the next step is performed.

Step 2. Initialize inner loop counter $s = 0$ and begin the inner loop of this step of the algorithm.

Inner loop. Form the test basis $\overline{I_s} = \{I_k, i_k\}$ and new affine subspace $\overline{H}_s = aff(x_i, i \in \overline{I}_s)$. Solve an auxiliary projection problem $\min_{z \in \overline{H}_s} \|z\|^2 = \|\overline{z^s}\|^2$.

If $\overline{z^s} \in \overline{Y}_s = co(x^i, i \in \overline{I}_s)$ then we put $I_{k+1} = \overline{I}_s$ and then move on to the next $(k+1)$-th iteration. Otherwise we set $u^\lambda = \lambda \overline{z}^s + (1-\lambda) z^k$ and find the maximum λ_s such as $u^{\lambda_s} \in \overline{Y}_s$. By construction, the point u^λ when $\lambda = \lambda_s$ belongs to the relative interior of a minimum face $\overline{Y}_{s+1}$ which is defined by a set of its extreme points

$x^i, i \in \overline{I}_{s+1}$, where $\overline{I}_{s+1}$ is a proper subset of $\overline{I}_s$ and $\sum_{i \in \overline{I}_{s+1}} \theta_i = 1,\ \theta_i > 0$ for $i \in \overline{I}_{s+1}$.

Increase the iteration counter of the inner loop $s = s + 1$ and repeat the iteration of the inner loop of step 2.

The finite convergence of this method is proved in [2] and its global "better than the geometric" rate of convergence – in [3].

4. Nested partitions method

In the case of a greater $(N > n+1)$ number of points in the set X an affine subspaces method described above is not applied. To solve the projection problem (1) with the condition $N > n+1$ a method of nested partitions is suggested. This method is based partly on the idea of a parallel projection method [6] but it includes: the first - dichotomy of the set X and the second - the fact that the partition of X can vary from iteration to iteration. The first fact makes it possible to apply the affine subspaces method and the second - to accelerate significantly the convergence of algorithm up to guarantee finiteness in contrast to [6].

Consider the dichotomy of X to Y_1 and Y_2, where $Y_1 = co(x^i, i \in I_1)$, $Y_2 = co(x^i, i \in I_2)$, $I_1 \cup I_2 = \Im$ and it is not necessarily $I_1 \cup I_2 = 0$.

In its original form a projection method [6] is as follows.

Step 1. (initialization of the method). Divide the set X into two subsets Y_1^0 and Y_2^0 in accordance with the criterion $\psi(x^i, p^k) = x^i p^0 - \|p^0\|^2$ where p^0 is the vector constructed as a convex combination of points of the set X. Then
$Y_1^0 = \{x^i \in X, \psi(x^i, p^k) \le 0\}$, $Y_2^0 = \{x^i \in X, \psi(x^i, p^k) > 0\}$.

Further, the k-th iteration of the method $(k = 0,1,2,...)$ consists of the following two steps.

Step 2. We form two sets $Y_1^k = \{Y_1^0, p^k\}$ and $Y_2^k = \{Y_2^0, p^k\}$ and for each of them solve the problem (1):

$$\|z_1^k\|^2 = \min_{z \in co(Y_1^k)} \|z\|^2, \quad \|z_2^k\|^2 = \min_{z \in co(Y_2^k)} \|z\|^2.$$

Step 3. Solve the problem $\min_{\lambda \in [0,1]} \|\lambda z_1^k + (1-\lambda) z_2^k\|^2 = \|p^{k+1}\|^2$ and check the condition of the completion of the algorithm

$$\psi(x^i, p^k) = x^i p^{k+1} - \|p^{k+1}\|^2 \ge 0 \text{ for } \forall x^i \in X. \tag{3}$$

If condition (3) is not satisfied then steps 2 and 3 are repeated.

Note that the sets Y_1^0 and Y_2^0 do not change by transition from one iteration to another. Asymptotic convergence of the algorithm was proved in [3], but computational experiments have shown very slow rate of convergence of the algorithm and therefore the question arose about its acceleration. The key to accelerating the convergence became the review of dichotomy of set X at each iteration. It turned out that changing the partition of

X significantly affects on the convergence of the algorithm, in particular, guarantees finite convergence. This method is called the nested partitions.
Nested partitions method is as follows.

Step 1. Construct a first approximation - the vector p^0 as a convex combination of points in X.

Step 2. Divide the set X into two subsets Y_1^k and Y_2^k as follows:

$$Y_1^k = \{x^i \in X, \psi(x^i, p^k) \le 0\}, \quad Y_2^k = \{x^i \in X, \psi(x^i, p^k) > 0\},$$

where $\psi(x^i, p^k)$ is defined as $\psi(x^i, p^k) = x^i p^k - \|p^k\|^2$. The index k $(k = 0,1,2,...)$ denotes the serial number of the iteration.

Step 3. Form two sets $\overline{Y}_1^k = \{Y_1^k, p^k\}$ and $\overline{Y}_2^k = \{Y_2^k, p^k\}$ and solve the problem (1) for each of them :

$$\|z_1^k\|^2 = \min_{z \in co(\overline{Y}_1^k)} \|z\|^2, \quad \|z_2^k\|^2 = \min_{z \in co(\overline{Y}_2^k)} \|z\|^2.$$

Step 4. Solve the problem

$$\min_{\lambda \in [0,1]} \|\lambda z_1^k + (1-\lambda) z_2^k\|^2 = \|p^{k+1}\|^2 \tag{4}$$

and check the condition of the completion of the algorithm:

$$\psi(x^i, p^k) = x^i p^{k+1} - \|p^{k+1}\|^2 \ge 0 \text{ for } \forall x^i \in X. \tag{5}$$

If condition (5) is not satisfied repeat steps 2, 3 and 4 of the algorithm. One iteration of nested partitions method includes steps 2, 3 and 4.

For the method of nested partitions the following theorem is true.

Theorem. Nested partitions method has finite convergence.

Proof. Let $M \subset \Im = \{1,2,...,N\}$ and $Y_M = co(\hat{x}^i, i \in M)$, $\|z^M\|^2 = \min_{z \in Y_M} \|z\|^2$,

$W_M = \{i : \hat{x}^i z^M < \|z^M\|^2\}$, $U_M = \{i : \hat{x}^i z^M = \|z^M\|^2\}$. It is clear that $W_M \cap U_M = 0$.

Let $\delta_M = \min_{i \in W_M} \{\|z^M\|^2 - \hat{x}^i z^M\}$ if $W_M \ne 0$ and $\delta_M = 0$ otherwise. If $W_M \ne 0$, z_M is the solution of the projection problem (2).

It is clear that $\delta_M > 0$ for $W_M \ne 0$ and by virtue of a finite number of M, there exists $\delta_M > 0$ such as $\delta_M \ge \delta > 0$ for M such as $W_M \ne 0$.

Lemma 1. There exists $\gamma > 0$ such that $\min_{z \in co(\hat{x}^i, i \in U_M \cup W_M)} \|z\|^2 \le \|z^M\|^2 - \gamma$ for all M for which $W_M \ne 0$.

Proof. This follows from elementary estimates. We have

$$\|\overline{z}^M\|^2 = \min_{z \in co(\hat{x}^i, i \in U_M \cup W_M)} \|z\|^2 \le \min_{z \in co(z^M, x^i, i \in W_M)} \|z\|^2. \tag{6}$$

This follows from the fact that $z_M \in co(\hat{x}^i, i \in U_M)$. Let $\hat{x}^M$ be such as

$$\hat{x}^M z^M - \|z^M\|^2 = \max_{i \in W_M} \{\hat{x}^i z^M - \|z^M\|^2\} = -\delta_M \le \delta < 0.$$

Then continuing the chain of inequalities (6) we obtain

$$\left\|\overline{z}^{M}\right\|^2 \le \min_{z\in co(z^M,x^M)} \|z\|^2 = \min_{\lambda\in[0,1]}\left\|z^M + \lambda(x^M - z^M)\right\|^2.$$

In this task, the minimum is reached when

$$\lambda_M = -z^M \frac{x^M - z^M}{\left\|x^M - z^M\right\|^2} = \frac{\delta_M}{\left\|x^M - z^M\right\|^2} > 0 \text{ and its value is}$$

$$\left\|z^M\right\|^2 - \frac{(z^M(x^M - z^M))^2}{\left\|x^M - z^M\right\|^2} \le \left\|z^M\right\|^2 - \frac{\delta_M^2}{2}\left(\left\|x^M\right\|^2 - \left\|z^M\right\|^2\right)^2 \le \left\|z^M\right\|^2 - \frac{\delta_M^2}{2}\Delta^2 \le \left\|z^M\right\|^2 - \frac{\delta^2}{2}\Delta^2,$$

where $\Delta = \max_{z\in X}\|z\| = \max_{i\in N}\left\|\hat{x}^i\right\|$.

When the algorithm the following sets are determined $U_k = \{i : \hat{x}^i z^k = \left\|z^k\right\|^2\}$, $W_k = \{i : \hat{x}^i z^k < \left\|z^k\right\|^2\}$ and $\left\|z^{k+1}\right\|^2 = \min_{z\in co(\hat{x}^i, i\in U_k\cup W_k)}\|z\|^2$ (or $i \in Z_k \supset U_k \cup W_k$), $\left\|z^k\right\|^2 = \min_{\hat{x}^i\in co(\hat{x}^i, i\in U_k)}\|z\|^2$.

Apply the lemma 1 for z^{k+1} and z^k: $0 < \left\|z^{k+1}\right\|^2 \le \left\|z^k\right\|^2 - \gamma$ with $\gamma \ge \frac{\delta^2}{2}\Delta^2$. It follows that after a finite number of steps the condition $W_k = 0$ will be satisfied. The theorem is proved.

5. Modification of the method of nested partitions

Nested partitions method allows various modifications. The basis for them is the following

Lemma 2. For the method of nested partitions $z_2^k = p^k$ is right.

Proof. For z_2^k the inequality $z_2^k x \ge \left\|z_2^k\right\|^2$ is right for any $x \in co(Y_2^k, p^k)$. However, by construction, $xp^k \ge \left\|p^k\right\|^2$ for any $x \in co(Y_2^k, p^k)$, and the last inequality holds with equality only if $x = p^k$. Due to the fact that z_2^k is a projection of the origin on the set $co(Y_2^k, p^k)$ there exist such λ_i^*, θ^* that

$$z_2^k = \sum_{i=1}^{n_k}\lambda_i^* x^i + \theta^* p^k,\ \ \sum_{i=1}^{n_k}\lambda_i^* + \theta^* = 1,\ \ \lambda_i^*, \theta^* \ge 0,\ \ n_k \subset I_2.$$

Multiplying vector z_2^k with p^k we get

$$z_2^k p^k = \sum_{i=1}^{n_k}\lambda_i^* x^i p^k + \theta^*\left\|p^k\right\|^2 \ge \sum_{i=1}^{n_k}\lambda_i^*\left\|p^k\right\|^2 + \theta^*\left\|p^k\right\|^2 = \left\|p^k\right\|^2.$$

We have $z_2^k p^k \ge \|p^k\|^2$ and $z_2^k x \ge \|z_2^k\|^2$ for any $x \in co(Y_2^k, p^k)$, including $x = p^k$, i.e. $z_2^k p^k \ge \|z_2^k\|^2$.

Add obtained two inequalities for $z_2^k p^k$: $\|p^k\|^2 + \|z_2^k\|^2 \le 2 z_2^k p^k$. Hence $\|p^k - z_2^k\| \le 0$ or $z_2^k = p^k$.

By the lemma 2 and the obvious inequality $\|z_1^k\|^2 \le \|z_2^k\|^2$ the number of calculations in this algorithm can be reduced by using projection z_1^k on $co(Y_1^k, p^k)$ from the previous iteration as the projection z_2^{k+1} on the set $co(Y_2^{k+1}, p^{k+1})$ for the next iteration.

A significant decrease in the number of iterations is possible if to include in the set Y_1^k some points of the set Y_2^k during a successive transition from one iteration to another. Inclusion criteria may be different which in turn affects on the total number of iterations. In particular, it is proposed to include in the set Y_1^k half points from the set Y_2^k having the lowest projection on the vector p^k.

6. Numerical experiment

For numerical experiments with algorithm initial simplex was randomly generated as a set of vectors with components which are independent uniformly distributed random variables. To generate a stress tests the scaling was used so that the last coordinate of each vector had a scale of measurement in 1000 times smaller than for the rest of the coordinates. More precisely, the initial data set was presented in each test as a set of m vectors $x = (x_1, x_2, ..., x_n)$ of the dimension n, which components are computed as

$$x_i = \begin{cases} 100(\xi_i - 0.5), & i = 1,2,...,n-1, \\ 0.1\xi_n, & \end{cases}$$

where ξ_i, $i = 1,2,...,n$ are independent uniformly distributed to [0,1] random variables.

The algorithm described in this paper has been implemented in the computer language Octave [7], freely available matrix-vector calculator, which is very convenient for such experiments.

Below there is a transcript of an experiment for the case of the simplex formed by the $m = 180$ vectors in space $n = 40$. The matrix X contains 180 vectors randomly generated in 40-dimensional space. The initial value for the generation of (40×180)-matrix X of initial data is given in the Octave by the command (seed, 31415292).

The experiment was conducted twice:

1) for the case of implementation of the algorithm of nested partitions method without modifications (simple algorithm);

2) for the case of inclusion to the set Y_1^k the points of the set Y_2^k in each iteration (modified algorithm).

Criterion for choosing the points into the modified set Y_1^k is as follows:

$$x^i p^k \le \|p^k\|^2 + \delta \text{, where } \delta = \frac{1}{10}\left(\max_{x^i \in Y_2^k} x^i p^k - \min_{x^i \in Y_2^k} x^i p^k\right).$$

The experimental results are presented in the following

Table. Convergence of the method of nested partitions

	Simple			Modified		
k	$\lvert Y_1^k \rvert$	$\lvert Y_2^k \rvert$	γ	$\lvert Y_1^k \rvert$	$\lvert Y_2^k \rvert$	γ
1	103	77	1.4259e-5	107	73	8.8047e-6
2	53	127	1.6226e-6	76	104	8.0904e-6
3	85	95	4.0042e-6	67	113	2.3769e-6
4	59	121	1.0308e-6	45	135	4.1006e-7
5	52	128	9.4053e-7	42	138	1.3291e-7
6	105	75	1.6121e-5	40	140	1.9523e-15
7	54	126	1.1471e-6			
8	47	133	3.9930e-7			
9	45	135	2.8230e-7			
10	42	138	8.4993e-8			
11	43	137	1.5126e-7			
12	40	140	1.9523e-15			

where k is the number of iteration, sets Y_1^k, Y_2^k are the dichotomy of X at the k-th iteration, $\lvert Y_1^k \rvert$, $\lvert Y_2^k \rvert$ are the powers of the sets Y_1^k and Y_2^k accordingly,

$\gamma = \dfrac{\|p^k\|^2 - \min_{x^i \in X} x^i p^k}{\|p^0\|^2 - \min_{x^i \in X} x^i p^0}$, p^0 is the center of gravity of the set X (step 1 of the algorithm), p^k is the solution of the problem (4) at the k-th iteration.

References

1. Kononov I.V. Methods for solving finite-variation inequalities. Kazan: DAS, 1998.
2. Nurminski E.A. Acceleration of iterative methods for the projection on the polyhedron// Investigated in Russia. 2005, 51-62 (http://zhurnal.ape.relarn.ru).
3. Nurminski E.A. Method of suitable affine subspaces for the projection on the simplex / / Journal of Computational Mathematics and Mathematical Physics, 2005. **45**, № 11, 1996-2004.
4. Zhang J., Xiu N. Some recent advances in projections-type methods for variation inequalities// Journal of Computational and Applied Mathematics. 2003. **152**, # 1, 2. 559-585.

5. Panin V.M., Skopetskiy V.V., Lavrina T.V. Models and methods finite variation inequalities// Cybernetics and system analysis. 2000. **6**. 47-64.
6. Nurminski E.A. A parallel method of projection into the convex envelope of a family of sets / / Izvestiya VUZov. Math. 2003. **12**, № 499, 78-82.
7. GNU Octave (http://www.octave.org).

D.V. Dolgy

Institute of Mathematics and Computer Sciences, Far Eastern Federal University, Vladivostok, Russia
Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail: *d_dol@mail.ru*

E.A. Nurminski

Institute of Mathematics and Computer Sciences, Far Eastern Federal University, Vladivostok, Russia
E-mail: *nurmi@dvo.ru*