

# A Conceptual Conjugate Epi-Projection Algorithm of Convex Optimization: Superlinear, Quadratic and Finite Convergence <sup>\*</sup>

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## Abstract

This paper considers a conceptual version of a convex optimization algorithm which is based on replacing a convex optimization problem with the root-finding problem for the approximate sub-differential mapping which is solved by repeated projection onto the epigraph of conjugate function. Whilst the projection problem is not exactly solvable in finite space-time it can be approximately solved up to arbitrary precision by simple iterative methods, which use linear support functions of the epigraph. It seems therefore useful to study computational characteristics of the idealized version of this algorithm when projection on the epigraph is computed precisely to estimate the potential benefits for such development. The key results of this study are that the conceptual algorithm attains super-linear rate of convergence in general convex case, the rate of convergence becomes quadratic for objective functions forming super-set of strongly convex functions, and convergence is finite when objective function has sharp minimum. In all cases convergence is global and does not require differentiability of the objective.

**Keywords:** convex optimization, conjugate function, approximate sub-differential, super-linear convergence, quadratic convergence, finite convergence, projection, epigraph

## Introduction

We consider a finite-dimensional nondifferentiable convex optimization problem (COP)

$$\min_{x \in E} f(x) = f_* = f(x^*), x^* \in X_*, \quad (1)$$

where  $E$  denotes a finite-dimensional space of primal variables and  $f : E \rightarrow \mathbb{R}$  is a finite convex function, not necessarily differentiable. As we are interested in computational issues related to solving (1) mainly we assume that this problem is solvable and has nonempty and bounded set of solutions  $X_*$ .

This problem enjoys a considerable popularity due to its important theoretical properties and numerous applications in large-scale structured optimization, discrete optimization, exact penalization in constrained optimization, and others. This led to the development of different algorithmic ideas, starting with the subgradient algorithm due to Shor (see [1] for the overview and references to earliest works) and followed by conjugate subgradient algorithms [2, 3] bundle methods [4], space dilatation and  $r$ -algorithms [5],  $\epsilon$ -subgradient methods [6, 7, 8],  $VU$ -methods [9], proximal point algorithms [10] and many others. These algorithms were widely used for solving many academical and practical problems,

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however only in a few cases the estimates for the rate of convergence were obtained. The most notable case is probably algorithms of proximal point family (PPA), which use the smooth approximation of the original COP by Moreau-Yosida regularization. In this case the superlinear rate of convergence was attained both for conceptual and implementable versions of PPA [11].

In this paper we suggest another algorithm with attractive rate of convergence at the conceptual level. It was considered by the author in [15]. and attracted our attention again due to positive computational experiments [16, 17] and some new computational ideas [18, 19]. It seems therefore useful to study computational characteristics of the idealized version of the algorithm when projection is computed precisely to gauge the potential benefits for such development. The new analysis of the conceptual algorithm showed that it does not only attains superlinear rate of convergence in quite general convex case, but the rate of convergence becomes quadratic for objective functions strictly convex in a vicinity of optimal solution, and the convergence is finite when objective has sharp minimum. In all cases the convergence is global and does not require different ability of the objective.

## 1 Notations and Preliminaries

Throughout the paper we use the following notations:  $E$  is a finite dimensional euclidean space of primal variables of any dimensionality. The inner product of vectors  $x, y$  from  $E$  is denoted as  $xy$  from  $E$ . The cone of non-negative vectors of  $E$  is denoted as  $E_+$ . The set of real numbers is denoted as  $\mathbb{R}$  and  $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ .

The norm in  $E$  is defined in a standard way:  $\|x\| = \sqrt{xx}$  and for  $X \subset E$   $\|X\| = \sup_{x \in X} \|x\|$ . This norm defines of course the standard topology on  $E$  with the common definitions of open and closed sets and closure and interior of subsets of  $E$ . The interior of a set  $X$  is denoted as  $\text{int}(X)$ .

The unit ball in  $E$  is denoted as  $B = \{x : \|x\| \leq 1\}$ . The support function of a set  $Z \subset E$  is denoted and defined as  $(Z)_x = \sup_{z \in Z} xz$ .

A vector of ones of a suitable dimensionality is denoted by  $e = (1, 1, \dots, 1)$ . A standard simplex  $\{x : x \geq 0, xe = 1\}$  with  $x \in E$ ,  $\dim(E) = n$  is denoted by  $\Delta_E$ .

We use the standard definitions of convex analysis (see f.i. [4]) related mainly to functions  $f : E \rightarrow \mathbb{R}_\infty$ : the domain of definition of  $f$  is the set  $\text{dom } f = \{x : f(x) < \infty\}$ , the epigraph  $\text{epi } f$  of a function  $f$  is a set  $\text{epi } f = \{(\mu, x) : \mu \geq f(x)\} \subset \mathbb{R}_\infty \times E$ .

Further on all functions are convex in a sense that their epigraphs are *convex* subsets of  $\mathbb{R}_\infty \times E$ .

**Definition 1** For a convex function  $f : E \rightarrow \mathbb{R}$  and fixed  $x \in E$  the set  $\partial f(x) = \{g : f(y) - f(x) \geq g(y - x) \text{ for all } y \in \text{dom } f\}$  is called a sub-differential of  $f$  at the point  $x$ .

The sub-differential of  $f$  is well-defined and is a closed bounded convex set for all  $x \in \text{int}(\text{dom } f)$ . At the boundary of  $\text{dom } f$  it may or may not exist. The sub-differential of  $f$  is also upper semi-continuous as a multi-function of  $x$  when exists.

**Definition 2** The directional derivative of a finite convex function  $f$  at point  $x$  in direction  $d$  is denoted and defined as

$$\partial f(x; d) = \lim_{\delta \rightarrow +0} (f(x + \delta d) - f(x))/\delta.$$

It is known from convex analysis that  $\partial f(x; d) = \sup_{g \in \partial f(x)} gd = (\partial f(x))_d$ .

**Definition 3** For a convex function  $f : E \rightarrow \mathbb{R}_\infty$  the function

$$f^*(g) = \sup_x \{gx - f(x)\} = (\text{epi } f)_{\bar{g}}, \text{ where } \bar{g} = (-1, g) \in \mathbb{R}_\infty \times E \quad (2)$$

is called a conjugate function of  $f$ .

The key result of convex analysis is that for a closed function  $f$  which epigraph  $\text{epi } f$  is a closed set

$$\sup_g \{gx - f^*(g)\} = (\text{epi } f^*)_{\bar{x}} = f(x), \quad (3)$$

where  $\bar{x} = (-1, x) \in \mathbb{R}_\infty \times E$ .

It is also easy to see that if  $(\text{epi } f^*)_{\bar{x}} = g_x x - f^*(g_x)$  then  $g_x \in \partial f(x)$  and the other way around: for  $\bar{g} = (-1, g)$  if  $(\text{epi } f)_{\bar{g}} = g x_g - f(x_g)$  then  $x_g \in \partial f^*(g)$ .

The trivial consequence of the Definition 3 is that  $f^*(0) = -\inf_x f(x)$  which is the key correspondence used by the conjugate epi-projection algorithm, considered further on. As the conjugate epi-projection algorithm operates in the conjugate space its convergence properties depend upon the properties of the conjugate function of the objective. Therefore we introduce some additional classes of primal functions to ensure the desired behavior of the conjugates.

**Definition 4** Convex function  $f$  is called sup-quadratic with respect to a point  $x \in \text{int}(\text{dom } f)$  if there exists a constant  $\tau > 0$  such that

$$f(y) - f(x) \geq g(y - x) + \frac{1}{2}\tau\|y - x\|^2 \quad (4)$$

for any  $g \in \partial f(x)$  and any  $y$ .

We will call  $\tau$  the sup-quadratic characteristic of  $f$  at  $x$ . Notice that strongly convex functions are sup-quadratic at any  $x$  from their domains, however a function  $f$ , sup-quadratic at some  $x$ , need not to be strongly convex.

A symmetric definition can be given for sub-quadratic functions.

**Definition 5** Convex function  $f$  is called sub-quadratic with respect to a point  $x \in \text{int}(\text{dom } f)$  if there exists a constant  $\tau > 0$  such that

$$f(y) - f(x) \leq g(y - x) + \frac{1}{2}\tau^{-1}\|y - x\|^2 \quad (5)$$

for any  $y \in \text{dom } f$  and some  $g \in \partial f(x)$ .

Notice that it follows from this definition that the function  $f$ , sub-quadratic at point  $x$  is in fact differentiable at this point. Of course not all functions differentiable at  $x$  are sub-quadratic.

Definitions 4 and 5 allow us to establish an important properties of conjugates functions for sup-quadratic primals.

**Lemma 1** Let  $f : E \rightarrow \mathbb{R}$  attains its minimum value  $f_*$  at the point  $x^*$  and  $f$  is sup-quadratic at point  $x^*$  with the positive sup-quadratic characteristic  $\tau$ . Then  $f^*(g)$  is sub-quadratic at  $g = 0$  with the corresponding sub-quadratic characteristic not lower then  $\tau^{-1}$ .

**Proof.** By definition for any  $x$

$$\frac{1}{2}\tau\|x^* - x\|^2 \leq f(x) - f_* = f(x) + f^*(0) \quad (6)$$

and hence

$$f^*(g) - f^*(0) = x_g g - (f(x_g) + f^*(0)) \leq x_g g - \frac{1}{2}\tau \|x^* - x_g\|^2 \quad (7)$$

for any  $x_g \in \partial f^*(g)$ . Hence

$$\begin{aligned} f^*(g) - f^*(0) &\leq x^* g + (x_g - x^*)g - \frac{1}{2}\tau \|x^* - x_g\|^2 \leq \\ &x^* g + \sup_z \{zg - \frac{1}{2}\tau \|z\|^2\} = x^* g + \frac{1}{2}\tau^{-1} \|g\|^2. \end{aligned} \quad (8)$$

■

Another interesting subclass of convex functions are those which have zero in the interior of the subdifferential at the solution  $x^*$  of a COP (1), that is  $0 \in \text{int}(\partial f(x^*))$ . This condition is also known as "sharp minimum" and extended further on in [12] and others. The special attraction of this case is that the known proximal method has then a finite termination [13] for such problems.

We notice now that the conjugate functions for objectives with sharp minimum have very simple behavior in the vicinity of zero which also guarantees the finite termination of the conjugate epi-projection algorithm as well.

**Lemma 2** *It solution  $x^*$  of (1) is such that  $0 \in \text{int}(\partial f(x^*))$  then there is  $\rho > 0$  such that  $f^*(g) = \sup_x \{gx - f(x)\} = gx^* - f(x^*)$  for  $\|g\| < \rho$ .*

**Proof.** The linearity of  $f^*(g)$  for  $g$  small enough follows from the fact that sharp minimum condition implies the existence of  $\rho > 0$  such that  $0 \in \partial(f(x^*) - gx^*)$  for any  $g \in \rho B$  and therefore for such  $g$

$$f^*(g) = \sup_x \{gx - f(x)\} = gx^* - f(x^*)$$

is a linear function of  $g$ . ■

For additional results on connections between sharp minimum and properties of conjugate functions see also [14].

## 2 Conjugate Epi-Projection Algorithm

As it was already mentioned the basic idea of the conjugate epi-projection algorithm consists in considering the convex problem (1) as the problem of computing the conjugate function of the objective at the origin:

$$f^*(0) = -\min_x f(x) = -f_* = \inf_{(0,\mu) \in \text{epi } f^*} \mu.$$

We suggest to use for computing  $f^*(0)$  the algorithm based on projection onto the epigraph  $\text{epi } f^*$ . This idea demonstrates some promises for effective solution of (1) and suggests some new computational ideas.

This version of the algorithm consists in execution of an infinite sequence of iterations, which generates the corresponding sequence of points  $\{(\xi_k, 0) \in \mathbb{R} \times E, k = 0, 1, \dots\}$  with  $\xi_k \rightarrow f^*(0)$  when  $k \rightarrow \infty$ . For each of these iterations it calls a subgradient oracle which for any  $x \in E$  computes  $f(x)$  and arbitrary  $g \in \partial f(x)$ . Also it requires solution of nonlinear projection problem which makes the algorithm strictly speaking unimplementable. However the analysis of the algorithm demonstrate its potential and can show the ways to its practical implementations. The principal details of the iteration of the conjugate epi-projection algorithm are given on the Fig. Algorithm 1. For better understanding of these two operations they are illustrated on the Fig 1, 2.

Convergence of the Algorithm 1 is confirmed by the following theorem.

**Data:** The convex function  $f : E \rightarrow \mathbb{R}$ , the epigraph  $\text{epi } f^*$ , the current iteration number  $k$  and the current approximation  $\xi_k \leq f^*(0)$ .

**Result:** The next approximation  $\xi_{k+1}$  such that  $\xi_k \leq \xi_{k+1} \leq f^*(0)$

Each iteration consists of two basic operations: **Project** and **Support-Update**

**Project.** Solve the projection problem of the point  $(\xi_k, 0)$  onto  $\text{epi } f^*$ :

$$\min_{(\xi, g) \in \text{epi } f^*} \{(\xi - \xi_k)^2 + \|g\|^2\} = (\xi_k^p - \xi_k)^2 + \|g_p^k\|^2$$

with the corresponding solution  $(\xi_k^p, g_p^k) = (f^*(g_p^k), g_p^k) \in \text{epi } f^*$ . We demonstrate in the analysis of the algorithm convergence that  $f^*(0) \geq \xi_k^p > \xi_k$  if  $\xi_k < f^*(0)$ .

**Support-Update** Compute support function of  $\text{epi } f^*$  with the support vector

$$z^k = -(\xi_k^p - \xi_k, g_p^k) \in \mathbb{R} \times E$$

$$\begin{aligned} (\text{epi } f^*)_{z^k} &= \sup_{(\mu, g) \in \text{epi } f^*} \{-(\xi_k^p - \xi_k)\mu + g_p^k g\} = \\ &(\xi_k^p - \xi_k) \sup_{(\mu, g) \in \text{epi } f^*} \left\{ -\mu + \frac{g_p^k}{(\xi_k^p - \xi_k)} g \right\} = (\xi_k^p - \xi_k) \sup_{(\mu, g) \in \text{epi } f^*} \{-\mu + x_p^k g\} = \\ &(\xi_k^p - \xi_k)(x_p^k g_p^k - f^*(g_p^k)) = (\xi_k^p - \xi_k)f(x_p^k), \end{aligned}$$

where  $x_p^k = g_p^k / (\xi_k^p - \xi_k)$ . Notice that as  $f$  is assumed to be a finite function this operation is well-defined.

Finally we update the approximate solution with  $\xi_{k+1}$  using the relationship

$$\bar{\xi}_{k+1} z^k = (\text{epi } f^*)_{z^k}, \text{ where } \bar{\xi}_{k+1} = (\xi_{k+1}, 0) \in \mathbb{R} \times E,$$

which actually amounts to  $\xi_{k+1} = -f(x_p^k)$ , increment iteration counter  $k \rightarrow k + 1$ , etc.

**Algorithm 1:** The basic iteration of the conceptual conjugate epi-projection algorithm

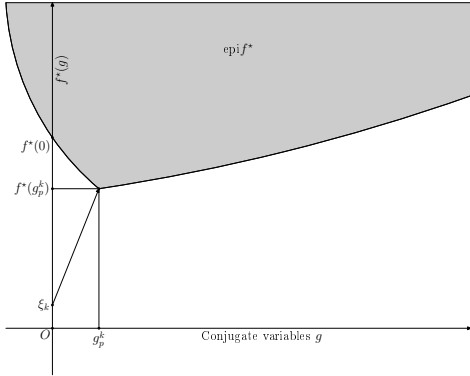


Figure 1: **Projection.** Solution of projection problem  $\min_{(\xi, g) \in \text{epi } f^*} \{(\xi - \xi_k)^2 + \|g\|^2\}$ .

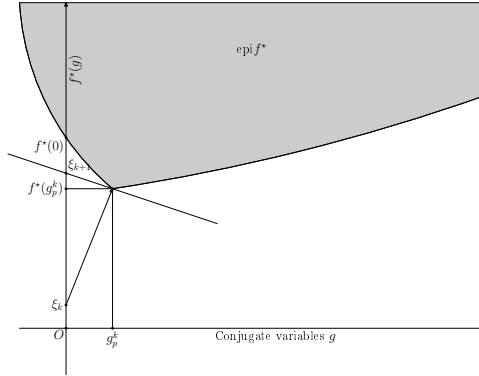


Figure 2: **Support-Update.** Compute support function  $v_k = (\text{epi } f^*)_{z^k}$  and update the approximate solution with  $\xi_{k+1} = v_k / (f^*(g_p^k) - \xi_k)$ .

**Theorem 1** Let  $f$  be a finite convex function with the finite minimum  $f_* = \min_x f(x) = -f^*(0)$  and  $\xi_k, k = 1, 2, \dots$  are defined by the Algorithm 1 with  $\xi_0 < f^*(0)$ . Then  $\lim_{k \rightarrow \infty} \xi_k = f^*(0) = -f_*$ .

**Proof.** Assume that on  $k$ -th iteration we have  $\xi_k < f^*(0)$  as the approximation of  $f^*(0)$ . According to Algorithm 1 to construct the next ( $k + 1$ -th) approximation  $\xi_{k+1}$  the point  $(\xi_k, 0) \in \mathbb{R} \times E$  is to be projected onto  $\text{epi } f^*$  first:

$$\min_{(\xi, g) \in \text{epi } f^*} \{(\xi - \xi_k)^2 + \|g\|^2\} = (\xi_k^p - \xi_k)^2 + \|g_p^k\|^2 \quad (9)$$

As a result the auxiliary point  $(\xi_k^p, g_p^k) = (f^*(g_p^k), g_p^k) \in \text{epi } f^*$  is obtained which satisfies optimality conditions

$$(f^*(g_p^k) - \xi_k)(\xi - \xi_k^p) + g_p^k(g - g_p^k) \geq 0 \quad (10)$$

for any  $(\xi, g) \in \text{epi } f^*$ .

It is easy to see that  $\xi_k^p > \xi_k$ . Indeed the opposite strict inequality  $\xi_k^p < \xi_k$  contradicts the optimality of  $(\xi_k^p, g_p^k)$  as in this case  $(\xi_k, g_p^k) = (\xi_k^p + (\xi_k - \xi_k^p), g_p^k) \in \text{epi } f^*$ , and

$$(\xi_k - \xi_k)^2 + \|g_p^k\|^2 < (\xi_k - \xi_k^p)^2 + \|g_p^k\|^2 = \min_{(\xi, g) \in \text{epi } f^*} \{(\xi_k - \xi)^2 + \|g\|^2\}.$$

If  $\xi_k^p = \xi_k$  then  $\mathbb{R} \times \{0\}$  is strictly separable from  $\text{epi } f^*$ :

$$\xi(\xi_k - \xi_k^p) + 0g_p^k = 0 < \|g_p^k\|^2 \leq \mu(\xi_k - \xi_k^p) + gg_p^k$$

for any  $(\mu, g) \in \text{epi } f^*$  as it follows from projection conditions. Hence  $0 \notin \text{dom}(f^*)$  which contradicts the assumptions of the theorem. According to Algorithm 1 the next approximation  $\xi_{k+1}$  is determined from the equality

$$(\xi_k^p - \xi_k)(\xi_{k+1} - \xi_k) - \|g_p^k\|^2 = (\xi_k^p - \xi_k)^2 + \|g_p^k\|^2$$

which gives the following expression for  $\xi_{k+1}$ :

$$\xi_{k+1} = \xi_k + \|g_p^k\|^2 / (\xi_k^p - \xi_k) \geq \xi_k,$$

and  $\xi_{k+1} = \xi_k$  if and only if  $g_p^k = 0$  which means that we already obtained the solution.

Repeating this operation we obtain the monotone sequence  $\xi_k, k = 0, 1, \dots$  such that

$$\xi_k \leq \xi_{k+1} \leq f^*(0), k = 0, 1, \dots$$

where inequalities turn into equalities only if either  $\xi_k = f^*(0)$  or  $\xi_{k+1} = f^*(0)$  which of course makes no difference. Under these conditions  $\lim_{k \rightarrow \infty} \xi_k = f^*(0)$  which proves the convergence of the algorithm 1. ■

Theorem 1 established the convergence of the Algorithm 1 under very general conditions, however to estimate the rates of convergence we need to derive more convenient estimates for decrease of convergence indicators. This is provided by the following lemma.

**Lemma 3** Let all assumptions of the theorem 1 be satisfied and  $\xi_k, k = 1, 2, \dots$  are defined by the Algorithm 1 with  $\xi_0 < -f_*$ . Then

$$f^*(0) - \xi_{k+1} \leq \|g_p^k\|(\partial f^*(g_p^k; z^k) - \partial f^*(0; z^k)), k = 1, 2, \dots \quad (11)$$

where  $z^k = g_p^k / \|g_p^k\|$ .

**Proof.** By construction  $\xi_{k+1} = -f(x_p^k) = f^*(g_p^k) - x_p^k g_p^k$ , where  $x_p^k = -g_p^k / (\xi_k^p - \xi_k)$ . Then

$$f^*(0) - \xi_{k+1} = f^*(0) - f^*(g_p^k) + x_p^k g_p^k \leq x_p^k g_p^k - x^* g_p^k$$

for any  $x^* \in \partial f^*(0) = X_*$ . Taking infimum of the right hand side with respect to  $x^* \in \partial f^*(0)$  obtain

$$\begin{aligned} f^*(0) - \xi_{k+1} &\leq x_p^k g_p^k - \partial f^*(0; g_p^k) \leq \sup_{x \in \partial f^*(g_p^k)} x g_p^k - \partial f^*(0; g_p^k) = \\ \partial f^*(g_p^k; g_p^k) - \partial f^*(0; g_p^k) &= \|g_p^k\| (\partial f^*(g_p^k; z^k) - \partial f^*(0; z^k)), k = 1, 2, \dots, \end{aligned} \quad (12)$$

where  $x_p^k = -g_p^k / (\xi_k^p - \xi_k)$  and where we used linear positive homogeneity of  $\partial f^*(\cdot; \cdot)$  with respect to its second argument. ■

The inequality (11) can be rewritten as

$$f^*(0) - \xi_{k+1} \leq \|g_p^k\| (\partial f^*(g_p^k; z^k) - \partial f^*(0; z^k)) = \|g_p^k\| \theta(g_p^k; z^k), \quad (13)$$

and depending on properties of the accuracy multiplier  $\theta(g_p^k; z^k)$  the convergence rates of Algorithm 1 will have different estimates.

First we establish super-linear rate of convergence of Algorithm 1 for the most general case of a finite objective function  $f$ .

**Theorem 2** *Let all assumptions of the theorem 1 be satisfied and  $\xi_k, k = 1, 2, \dots$  are defined by the Algorithm 1 with  $\xi_0 < -f_*$ . Then  $f^*(0) - \xi_{k+1} \leq \lambda_k (f^*(0) - \xi_k)$  with  $\lambda_k \rightarrow 0$  when  $k \rightarrow \infty$ .*

**Proof.** For the finite  $f$  and bounded nonempty  $X_*$  in the problem (1) the conjugate function  $f^*$  has nonempty  $\text{dom}(f^*)$  and  $0 \in \text{int}(\text{dom}(f^*))$ .

Then due to convergence of Algorithm 1  $g_p^k \rightarrow 0$  when  $k \rightarrow \infty$ . In the notations of Algorithm 1  $f^*(2g_p^k) - f^*(g_p^k) \geq f^*(g_p^k) - f^*(0)$  by convexity and hence

$$g_p^k x^* \leq f^*(g_p^k) - f^*(0) \leq f^*(2g_p^k) - f^*(g_p^k) \leq p^k g_p^k$$

for any  $x^* \in \partial f^*(0)$  and  $p^k \in \partial f^*(2g_p^k)$ .

After division by  $\|g_p^k\| > 0$  it gives

$$z^k x^* \leq p^k z^k \quad (14)$$

where  $z^k = g_p^k / \|g_p^k\|$ .

Taking supremum of the left hand side of the inequality (14) with respect to  $x^* \in \partial f^*(0)$  obtain

$$\partial f^*(0; z^k) \leq z^k p^k$$

Assuming that  $z^k \rightarrow z^\bullet, p^k \rightarrow p^\bullet$  when  $k \rightarrow \infty$  and  $g_p^k \rightarrow 0$  according to Theorem 1 obtain

$$\partial f^*(0; z^\bullet) \leq z^\bullet p^\bullet.$$

As  $p^\bullet \in \partial f^*(0)$  by upper semi-continuity of the sub-differential mapping  $\partial f^*(\cdot)$

$$\partial f^*(0; z^\bullet) \leq z^\bullet p^\bullet \leq \sup_{p \in \partial f^*(0)} z^\bullet p = \partial f^*(0; z^\bullet)$$

which implies that  $z^k p^k \rightarrow \partial f^*(0; z^\bullet)$  when  $k \rightarrow \infty$  or

$$\partial f^*(g_p^k; z^k) - \partial f^*(0; z^k) = \theta(g_p^k; z^k) \rightarrow 0 \quad (15)$$

when  $k \rightarrow \infty$ . Putting everything together we obtain

$$f^*(0) - \xi_{k+1} \leq \theta(g_p^k; z^k) \|g_p^k\|, \quad \theta(g_p^k; z^k) \rightarrow +0, \quad \text{when } k \rightarrow \infty.$$

As  $x^k = -g_p^k / (f^*(g_p^k) - \xi_k)$  than due to upper semi-continuity of  $\partial f^*$

$$\|g_p^k\| = \|x^k\| (f^*(g_p^k) - \xi_k) \leq 2\|X_\star\| (f^*(g_p^k) - \xi_k) \leq 2\|X_\star\| (f^*(0) - \xi_k) \quad (16)$$

and consequently

$$f^*(0) - \xi_{k+1} \leq 2\theta(g_p^k, z^k)\|X_\star\| (f^*(0) - \xi_k) = \lambda_k (f^*(0) - \xi_k)$$

with  $\lambda_k \rightarrow 0$  when  $k \rightarrow \infty$ . ■

Next we consider the problem (1) with sup-quadratic objective function  $f$ .

**Theorem 3** *Let objective function  $f$  in problem (1) is locally sup-quadratic with sup-quadratic characteristic  $\tau$  and  $\xi_k, k = 1, 2, \dots$  are defined by the Algorithm 1 with  $\xi_0 < -f_\star$ . Then  $\lim_{k \rightarrow \infty} \xi_k = f^*(0)$  (Algorithm 1 converges) and for  $k$  large enough  $f^*(0) - \xi_{k+1} \leq \tau^{-1} (f^*(0) - \xi_k)^2$  (that is convergence is quadratic).*

**Proof.** It follows from local sup-quadratic behavior of  $f$  that  $f^*$  is differentiable in some neighborhood  $U$  of 0. Therefore the subdifferentials of  $f^*(g)$  are singletons and we can consider  $\partial f^*(g)$  as just a vector. It follows from sup-quadratic behavior of  $f$  that  $\|\partial f^*(g) - \partial f^*(0)\| \leq \tau^{-1}\|g\|$ . Consequently

$$\partial f^*(g_p^k; z^k) - \partial f^*(0; z^k) = \partial f^*(g_p^k)z^k - \partial f^*(0)z^k \leq \|\partial f^*(g_p^k) - \partial f^*(0)\| \leq \tau^{-1}\|g_p^k\| \quad (17)$$

gives exactly  $f^*(0) - \xi_{k+1} \leq C(f^*(0) - \xi_k)^2$  with  $C = \tau^{-1}$ . ■

Finally we consider the case of a sharp minimum in (1), namely that  $0 \in \text{int}(\partial f(x^*))$ .

**Theorem 4** *Let the objective function of (1) has a sharp minimum at solution point  $x^*$ , all assumptions of the theorem 1 are satisfied and  $\xi_k, k = 1, 2, \dots$  are defined by the Algorithm 1 with  $\xi_0 < -f_\star$ . Then there exists  $k^*$  such that  $\xi_{k^*} = f^*(0) = -f_\star$ .*

**Proof.** According to Lemma 2 under conditions of sharp minimum there is a neighborhood  $U$  of  $g = 0$  such that

$$f^*(g) = \sup_x \{gx - f(x)\} = gx^* - f_\star$$

is a linear function of  $g$  in  $U$ .

By Theorem 1  $g_p^k \in U$  for  $k$  large enough and let  $k^* - 1$  is the first such index that  $g_p^{k^*-1} \in U$ . Then

$$\partial f^*(g_p^{k^*-1}; g_p^{k^*-1}) = g_p^{k^*-1}x^* = \partial f^*(0; g_p^{k^*-1})$$

and hence

$$0 \leq \xi_{k^*} - f^*(0) \leq \partial f^*(g_p^{k^*-1}; g_p^{k^*-1}) - \partial f^*(0; g_p^{k^*-1}) = 0$$

and Algorithm 1 terminates. ■

## Conclusion

The conceptual version of the dual epi-projection algorithm has promising computational properties which make it a viable candidate for developing implementable versions. First of all it guarantees global super-linear convergence to the optimum in any solvable COP. Second, it provides quadratic convergence and even finite termination without any changes in the algorithm for quite common types of COPs: sup-quadratic, which strictly contain strongly convex, and COPs with sharp minimum. It is worth to notice that the algorithm is absolutely parameter-free, use the first-order subgradient oracle



only, and requires no specific knowledge of any specific characteristics of COP, like Lipschitz constants, strong convexity parameter or close enough initial approximation.

The implementation perspectives for the algorithm depend upon the possibility to produce practical version of the projection operator on  $\text{epi } f^*$ . From the theoretical point of view it is easy to derive accuracy estimates for its termination so it can be finitely solved for any required accuracy. It can be used to preserve the overall rates of convergence in terms of Algorithm 1 iterations, however the resulting computational complexity requires further investigations.

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