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Projection onto Polyhedra in Outer Representation

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Abstract—The projection of the origin onto an n-dimensional polyhedron defined by a system of m inequalities is reduced to a sequence of projection problems onto a one-parameter family of shifts of a polyhedron with at most $m + 1$ vertices in $n + 1$ dimensions. The problem under study is transformed into the projection onto a convex polyhedral cone with m extreme rays, which considerably simplifies the solution to an equivalent problem and reduces it to a single projection operation. Numerical results obtained for random polyhedra of high dimensions are presented.

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Let $X = \{x: Ax \le b\}$ be a nonempty convex polyhedral subset of the n-dimensional Euclidean space E_n defined by a system of m inequalities, where A is an $(m \times n)$-matrix and b is an m-vector. Consider the problem

$$\min_{x \in X} \frac{1}{2}\|x\|^2 = \frac{1}{2}\|x^*\|^2; \tag{1}$$

i.e., we search for the least norm point x^* in this set. To avoid triviality, it is assumed that $0 \notin X$. Problem (1) arises in pattern recognition and data processing and is used in various computational procedures.

In this paper, we propose an algorithm for its solution that is based on the procedure of [1, 2] for projecting onto a polyhedron in internal representation, i.e., defined as the convex hull of its extreme points. A global better-than-linear estimate for the convergence rate of this procedure was obtained in [2], and numerical experiments have shown that it is quite efficient for problems of fairly large dimensions.

In principle, the original polyhedron X can be initially represented as the convex hull of its extreme points. However, the direct application of the algorithm [2] to this representation is inexpedient for several reasons. First, the number of extreme points in this polyhedron is exponential in terms of the number of constraints (m) and is polynomial with an exponent of order m in terms of the number of variables (n). Second, the extreme points of X are given implicitly and their selection for the algorithm of [2] requires solving an auxiliary linear programming problem. Despite the success of polynomial algorithms in linear programming, this problem remains difficult. Moreover, these algorithms are poorly suited for series of modified problems, which is an important factor in the case under study. The classical simplex method has an exponential complexity bound and may lead to infinite iterations.

These circumstances motivated the study of alternative approaches. As a result, problem (1) was reduced to a series of projections onto one-parameter shifts of a dual polyhedron with $(m + 1)$ vertices in an n-dimensional space. A further analysis showed that it is sufficient to obtain such a projection for only one shift of the polyhedral convex cone with m generatrices.

1. NOTATION AND PRELIMINARY RESULTS

All the vectors below are assumed to belong to finite-dimensional Euclidean spaces of suitable dimensions. Given a set of vectors $D = \{a^1, \ldots, a^m\}$, we denote by $\mathrm{aff}\{D\}$ its affine hull

$$\mathrm{aff}\{D\} = \left\{ x = \sum_{i=1}^{m} \mu_i a^i, \sum_{i=1}^{m} \mu_i = 1 \right\};$$

by Co$\{D\}$, its conical hull

$$\mathrm{Co}\{D\} = \left\{ x = \sum_{i=1}^{m} \mu_i a^i, \mu_i \ge 0 \right\};$$

and by co$\{D\}$ its convex hull

$$\mathrm{co}\{D\} = \left\{ x = \sum_{i=1}^{m} \mu_i a^i, \mu_i \ge 0, \sum_{i=1}^{m} \mu_i = 1 \right\}.$$

The scalar product of two vectors x and y is designated as xy. The case when a vector is multiplied by a scalar is usually clear from the context.

Let e be a vector of suitable dimension consisting of ones. Denote by Δ_m the m-dimensional standard simplex

$$\Delta_m = \left\{ s = (s_1, \ldots, s_m) : \sum_{i=1}^{m} s_i = se = 1, s_i \ge 0, i = 1, 2, \ldots, m \right\}.$$

We show that problem (1) can be reduced to an equivalent unconditional optimization problem with a nonsmooth exact penalty.

Lemma 1. *There exists* $\Gamma > 0$ *such that, for all* $\gamma \ge \Gamma$, *problem* (1) *is equivalent to*

$$\min\left\{ \frac{1}{2}\|x\|^2 + \gamma|Ax - b|_\infty^+ \right\}, \tag{2}$$

where $|Ax - b|_\infty^+ = \max\{0, \max_{i = 1, 2, \ldots, n} (Ax - b)_i\}$ *and* $a(Ax - b)_i$ *is the ith coordinate of the vector* $Ax - b$.

Proof. The optimality conditions for problem (1) have the form

$$x^* + u^*A = 0, \tag{3}$$

where $u^* \ge 0$ are optimal dual variables. In the nontrivial case, $u^*e = \Gamma > 0$. Conditions (3) are rewritten as

$$x^* + \Gamma\mu^*A = 0 \tag{4}$$

with $\mu^* = u^*/(u^*e) \in \Delta_m$, and we set $\psi(x) = |Ax - b|_\infty^+$. Since x^* is feasible, we have $\psi(x^*) = 0$.

The complementarity conditions imply that $\mu^*(Ax^* - b) = 0$. Therefore, for arbitrary x, we have

$$\psi(x) - \psi(x^*) = |Ax - b|_\infty^+ = \max\{0, \max_{i = 1, 2, \ldots, n} (Ax - b)_i\} \ge \mu^*(Ax - b) - \mu^*(Ax^* - b) = \mu^*A(x - x^*);$$

therefore, $\mu^*A \in \partial\psi(x^*)$. Consequently, condition (4) can be rewritten as

$$0 \in \partial\left\{ \frac{1}{2}\|x^*\|^2 + \Gamma\psi(x^*) \right\}$$

and, accordingly, x^* minimizes the weighted objective function of problem (2). Since the solutions to problems (1) and (2) are unique, the converse is also true.

For $\gamma > \Gamma$, it follows that

$$\frac{1}{2}\|x^*\|^2 + \gamma\psi(x^*) = \frac{1}{2}\|x^*\|^2 + \Gamma\psi(x^*) \le \frac{1}{2}\|x\|^2 + \Gamma\psi(x) \le \frac{1}{2}\|x\|^2 + \gamma\psi(x),$$

which proves the optimality of x^* for all $\gamma > \Gamma$. The proof is complete.

For further constructions, we add one more coordinate to the vector x, denoting the result by $\bar{x} = (x, x_{n+1})$, and define the augmented matrix $\bar{A} = \|A|b\|$. Then, $x \in X$ is equivalent to $\bar{x} \in \bar{X} = \{\bar{x} : \bar{A}\bar{x} \le 0,$

$x_{n+1} = \bar{x}e^{n+1} = -1\}$, where $e^{n+1} = (0, \ldots, 0, 1)$ is the $(n+1)$th basis vector. Moreover,

$$|Ax - b|_\infty^+ = |\bar{A}\bar{x}|_\infty^+ = \max\{0, \max_{\lambda \in \Delta_m} \lambda\bar{A}\bar{x}\} = \max_{\lambda \in \Delta_m^0} \lambda\bar{A}\bar{x}, \tag{5}$$

where $\Delta_m^0 = \{\lambda : \Sigma_{i=1}^m \lambda_i \le 1, \lambda_i \ge 0, i = 1, 2, \ldots, m\}$.

Theorem 1. *Let $u \in R$ be a scalar variable, $\bar{A}^i$ $(i = 1, 2, \ldots, m)$ be the row vectors of the matrix $\bar{A}$, $\gamma > \Gamma$, and*

$$D_\gamma(u) = \mathrm{co}\{0, \gamma\bar{a}^i, i = 1, 2, \ldots, m\} + ue^{n+1} = D_\gamma + ue^{n+1}.$$

Then,

$$\min_{x \in X} \frac{1}{2}\|x\|^2 = \frac{1}{2}\|x^*\|^2 = -\min_u\{\phi_\gamma(u) - u\}, \tag{6}$$

where

$$\phi_\gamma(u) = \min_{\bar{x} \in D_\gamma(u)} \frac{1}{2}\|\bar{x}\|^2.$$

Proof. By Lemma 1 and (5), problem (2) can be transformed into

$$\begin{aligned}\min\left\{\frac{1}{2}\|x\|^2 + \gamma|Ax - b|_\infty^+\right\} &= \min_{\bar{x}_{n+1}+1=0}\left\{\frac{1}{2}(\|\bar{x}\|^2 - 1) + \gamma \max_{\lambda \in \Delta_{n+1}^0} \lambda\bar{A}\bar{x}\right\} \\ = \min_{\bar{x}_{n+1}+1=0}\left\{\frac{1}{2}\|\bar{x}\|^2 + (\gamma D)_{\bar{x}}\right\} - \frac{1}{2} &= \min_{\bar{x}_{n+1}+1=0}\left\{\frac{1}{2}\|\bar{x}\|^2 + \gamma(D)_{\bar{x}}\right\} - \frac{1}{2},\end{aligned} \tag{7}$$

where $D = \{d : d = \lambda\bar{A}, \lambda \in \Delta_m^0\}$ and $(D)_{\bar{x}} = \sup_{d \in D_\gamma} d\bar{x}$ is the support function of D. It is easy to see that D can be represented as the convex hull of its extreme points, which correspond to the rows $\bar{A}_i$ of $\bar{A}$ $(i = 1, 2, \ldots, m)$ supplemented by a zero vector:

$$D = \mathrm{co}\{0, \bar{A}_i, i = 1, 2, \ldots, m\}.$$

Note that the number of extreme points of D is only one more than the number of constraints.

The last problem in (7) can be transformed by applying the Lagrangian relaxation of the only constraint in

$$\begin{aligned}&\min_x \max_u\left\{\frac{1}{2}\|\bar{x}\|^2 + (D_\gamma)_{\bar{x}} + u(\bar{x}e^{n+1} + 1)\right\} \\ &= \max_u\left\{u + \min_{\bar{x}}\left\{\frac{1}{2}\|\bar{x}\|^2 + (D_\gamma + ue^{n+1})_{\bar{x}}\right\}\right\},\end{aligned} \tag{8}$$

where $e^{n+1} = (0, \ldots, 0, 1)$ and the sum $\gamma D + ue^{n+1}$ denotes the set $\{\gamma d + ue^{n+1}, d \in D\}$, i.e., the shift of γD by u in the direction of e^{n+1}.

It was noted in [3] that, for an arbitrary convex closed set C,

$$-\min_{\bar{x}}\left\{\frac{1}{2}\|\bar{x}\|^2 + (C)_{\bar{x}}\right\} = \min_{\bar{x} \in C} \frac{1}{2}\|\bar{x}\|^2$$

(see also [4, Section 3.2], although it was not explicitly indicated therein). Define

$$\phi_\gamma(u) = \min_{\bar{x} \in D_\gamma(u)} \frac{1}{2}\|\bar{x}\|^2, \tag{9}$$

where

$$D_\gamma(u) = \gamma D + ue^{n+1} = \text{co}\{ue^{n+1}, \gamma\bar{A}_i + ue^{n+1}, i = 1, 2, \dots, m\}.$$

Applying this relation to (8), we finally obtain the equivalence

$$\min_{x \in X} \frac{1}{2}\|x\|^2 = -\min_u \{\phi_\gamma(u) - u\}.$$

The usefulness of (6) depends on the properties of ϕ_γ and on how effectively it is computable.

2. PROPERTIES OF THE MINIMUM-DISTANCE FUNCTION

We examine the properties of $\phi_\gamma(u)$, which is called the minimum-distance function, and propose a finite algorithm for solving the problem

$$\min_u \{\phi_\gamma(u) - u\} = \phi_\gamma(u^*) - u^*. \quad (10)$$

Lemma 2. *The function ϕ_γ is convex, smooth, and piecewise quadratic.*

Proof. Let $\lambda \in [0, 1]$ and $\nu_\lambda = \lambda\nu' + (1 - \lambda)\nu''$. For $\lambda \in [0, 1]$ and an arbitrary convex closed set B, we have $\lambda B + (1 - \lambda)B = B$. Therefore,

$$\phi_\gamma(\nu_\lambda) = \min_{z \in D(\nu_\lambda)} \frac{1}{2}\|z\|^2 = \min_{z \in \lambda(D + \nu' e^{n+1}) + (1-\lambda)(D + \nu'' e^{n+1})} \frac{1}{2}\|z\|^2$$

$$= \min_{\substack{z' \in D + \nu' e^{n+1} \\ z'' \in D + \nu'' e^{n+1}}} \frac{1}{2}\|\lambda z' + (1 - \lambda)z''\|^2 \le \min_{\substack{z' \in D + \nu' e^{n+1} \\ z'' \in D + \nu'' e^{n+1}}} \frac{1}{2}\{\lambda\|z'\|^2 + (1 - \lambda)\|z''\|^2\}$$

$$= \lambda \min_{z' \in D + \nu' e^{n+1}} \frac{1}{2}\|z'\|^2 + (1 - \lambda) \min_{z' \in D + \nu'' e^{n+1}} \frac{1}{2}\|z''\|^2 = \lambda\phi_\gamma(\nu') + (1 - \lambda)\phi_\gamma(\nu''),$$

which proves the convexity of ϕ_γ. The smoothness of $\phi_\gamma(\cdot)$ follows from the uniqueness of the solution to problem (9). The piecewise quadratic property follows from the fact that its solution is a piecewise linear function of u.

For the derivative of $\phi_\gamma(u)$ we have $\phi'_\gamma(u) = \bar{x}^*(u)e^{n+1} = \bar{x}^*_{n+1}(u)$, where $\bar{x}^*(u)$ solves problem (9) and $\bar{x}^*_{n+1}(u)$ is the $(n + 1)$th coordinate of this vector.

In fact, $\phi'_\gamma(u)$ can be computed without using any additional operation in comparison with the computation of $\phi_\gamma(u)$. In these conditions, it is easy to organize a dichotomy process such that the uncertainty interval for the solution to problem (10) decreases in geometric progression with a common ratio of 0.5, which does not depend on the parameters of the problem. Since ϕ'_γ is finite piecewise linear, u^* can also easily be computed *exactly* for a sufficiently small interval $[u_1, u_r]$ satisfying $u_l < u_r$, $\phi'_\gamma(u_l) < 1$, and $\phi'_\gamma(u_r) > 1$ and having no more than two linearity segments for ϕ'_γ. Moreover, despite the potentially exponential number of segments where ϕ_γ is piecewise quadratic, the number of iterations in the algorithm for solving problem (10) is linear in terms of the number of constraints and logarithmic in terms of the number of variables. The polynomial character of the entire computational procedure depends on the computational complexity of $\phi_\gamma(u)$. According to [6], the quadratic programming problem with a fixed number of constraints is computable in time, that is, polynomial in the length of the input, which is constructively demonstrated by applying the linear-in-the-input-length ellipsoid method with $O((n + m)^4L)$ complexity, where L is the length of integer input data in binary form, n is the number of variables, and m is the number of constraints in the problem.

Computationally, the solution procedure for problem (10) can be considerably simplified by proceeding from the bounded sets $D_\gamma(u)$ to the cone Co$\{D\}$. It turns out that, despite its extremal method of specification, the function

$$\phi(u) = \min_{x \in \text{Co}(D) + ue^{n+1}} \|x\|^2, \quad (11)$$

which is a direct analogue of $\phi_\gamma(u)$ in (9), becomes the simplest quadratic polynomial in $\phi(u) = \alpha^2u^2$.

Indeed, let $x^*(u) = z^*(u) + ue^{n+1}$ be a solution to problem (11) with $z^*(u) \in \mathrm{Co}(D)$. Then, by the optimality conditions, for any $z \in \mathrm{Co}(D)$, we have

$$(z + ue^{n+1})(z^*(u) + ue^{n+1}) \geq \|z^*(u) + ue^{n+1}\|^2.$$

Multiplying this inequality by τ^2, where $\tau = u'/u > 0$ with $u' > 0$, yields

$$(\tau z + \tau ue^{n+1})(\tau z^*(u) + \tau ue^{n+1}) = (\tau z + u'e^{n+1})(\tau z^*(u) + u'e^{n+1})$$
$$\geq \|\tau z^*(u) + \tau ue^{n+1}\|^2 = \|\tau z^*(u) + u'e^{n+1}\|^2.$$

Since $\tau\mathrm{Co}(D) = \mathrm{Co}(D)$,

$$(z + u'e^{n+1})(\tau z^*(u) + u'e^{n+1}) \geq \|\tau z^*(u) + u'e^{n+1}\|^2$$

for any $z \in \mathrm{Co}(D)$. Therefore, after replacing u with u', the solution to problem (11) becomes $x^*(u') = (u'/u)z^*(u) + u'e^{n+1}$. Hence, $x^*(u) = u(w^* + e^{n+1})$ for some $w^* \in \mathrm{Co}D$ independent of u and, accordingly, $\phi(u) = \|w^* + e^{n+1}\|^2u^2 = \alpha^2u^2$, which was to be proved. Note also that $\phi(u) \leq \phi_\gamma(u)$ for any γ and u.

To justify the transition to conical hulls, we prove several simple assertions.

Lemma 3. *If* $\phi_\gamma(\bar{u}) > \phi(\bar{u})$ *for some* $\bar{u}$, *then* $\phi_\gamma(u) > \phi(u)$ *for all* $u > \bar{u}$.

Proof. Let $\kappa > 1$ and $u = \kappa\bar{u} > \bar{u}$. Then, $\mathrm{co}\{\kappa D_\gamma\} \supset \mathrm{co}\{D_\gamma\}$. Therefore,

$$\phi_\gamma(\bar{u}) = \min_{d \in \mathrm{co}\{D_\gamma\}} \frac{1}{2}\|d + \bar{u}e\|^2 = \frac{1}{\kappa^2} \min_{d \in \mathrm{co}\{D_\gamma\}} \frac{1}{2}\|\kappa d + \kappa\bar{u}e\|^2$$
$$= \frac{1}{\kappa^2} \min_{d \in \mathrm{co}\{\kappa D_\gamma\}} \frac{1}{2}\|d + ue\|^2 \leq \frac{1}{\kappa^2} \min_{d \in \mathrm{co}\{D_\gamma\}} \frac{1}{2}\|d + ue\|^2 = \frac{1}{\kappa^2}\phi_\gamma(u);$$

i.e.,

$$\phi_\gamma(u) \geq \kappa^2\phi_\gamma(\bar{u}) > \kappa^2\phi(\bar{u}) = \phi(u),$$

which was to be proved.

Corollary 1. If $\phi_\gamma(\bar{u}) = \phi(\bar{u})$ for some $\bar{u}$, then $\phi_\gamma(u) = \phi(u)$ for all $u \leq \bar{u}$.

This statement implies the following result.

Lemma 4. *For any* $\bar{u}$, *there exists* Γ *such that* $\phi_\gamma(u) = \phi(u)$ *for all* $u \leq \bar{u}$, *where* $\gamma > \Gamma$.

Proof. Let $z^*_{\bar{u}}$ be the solution to problem (11) at $u = \bar{u}$, which obviously exists. Let Γ be such that $z^*_{\bar{u}} \in \Gamma\mathrm{co}\{D\} = \mathrm{co}\{D_\Gamma\} \subset \mathrm{co}\{D_\gamma\}$ for all $\gamma > \Gamma$. Obviously, $\phi_\gamma(\bar{u}) = \phi(\bar{u})$. By Corollary 1 to Lemma 3, this proves the lemma.

The lemmas stated above give the final expression for the solution to problem (1).

Theorem 2. *It is true that*

$$\min_{x \in X} \frac{1}{2}\|x\|^2 = \frac{1}{2}\phi(1)[\phi(1) - 1]. \tag{12}$$

Proof. Indeed, $\phi(u)$ can be written as $\phi(u) = \alpha^2u^2 = \phi(1)u^2$, which implies that the solution to the problem

$$\phi_* = \min_u[\phi(u) - u] = \min_u[\phi(1)u^2 - u] \tag{13}$$

has the form $u_* = \phi(1)/2$. Therefore, $\phi_* = \phi(1)[\phi(1) - 1]/2$. Set $\bar{u} = 2u_*$. Defining Γ as the maximum of those delivered by Lemmas 1 and 4 and assuming that $\gamma > \Gamma$, we obtain

$$\min_{x \in X} \frac{1}{2}\|x\|^2 = \min_u[\phi_\gamma(u) - u] = \min\{\min_{u \in [0, \bar{u}]}[\phi_\gamma(u) - u], \min_{u \geq \bar{u}}[\phi_\gamma(u) - u]\}$$
$$= \min\{\min_{u \in [0, \bar{u}]}[\phi(u) - u], \min_{u \geq \bar{u}}[\phi_\gamma(u) - u]\}.$$

It is always true that $\phi_\gamma(u) \ge \phi(u)$ for $u > \bar{u}$. Therefore,

$$\min_{u \ge \bar{u}}[\phi_\gamma(u) - u] \ge \min_{u \ge \bar{u}}[\phi(u) - u] > \min_{u \in [0, \bar{u}]}[\phi(u) - u]$$

and, hence,

$$\min_{x \in X}\frac{1}{2}\|x\|^2 = \min_{u \in [0, \bar{u}]}[\phi(u) - u] = \min_{u}[\phi(u) - u] = \phi_*,$$

which was to be shown.

With the transition to the conical hulls of the rows $\bar{A}_i$ of $\bar{A}$ $(i = 1, 2, \dots, m)$, the solution of problem (10) is considerably simplified, but we now need efficient algorithms for the projection of $\min\limits_{x \in \mathrm{Co}(D) + ue^{n+1}}\|x\|^2$ onto Co(D). A possible option is to apply the modification of the algorithm of [2] that is described below.

3. PROJECTION ONTO A CONE DEFINED BY GENERATRICES

Since the specific character of the shift vector is of no matter, we consider the general problem

$$\min_{z \in \mathrm{Co}(D) + a}\|z\|^2 = \|z^*\|^2 = \min_{x \in \mathrm{Co}(D)}\|x + a\|^2 = \|x^* + a\|^2, \tag{14}$$

where a is a shift vector and $\mathrm{Co}\{D\} = \mathrm{Co}(a^1, \dots, a^m)$ is the convex cone generated by the system of vectors $D = \{a^1, \dots, a^m\}$. We also consider the conical hulls of subsets of D, which are subcones of $\mathrm{Co}\{D\}$. In this context, we need to define a suitable subcone by analogy with the definition of a suitable subsimplex in [2].

Definition. The set $\mathrm{Co}\{D'\}$ with $D' \subset D$ is called a suitable cone if

$$\min_{z \in a + \mathrm{aff}\{D'\}}\|z\|^2 = \min_{z \in a + \mathrm{Co}\{D'\}}\|z\|^2. \tag{15}$$

Note that any one-dimensional cone $\mathrm{Co}\{a^i\}$ with $aa^i \le 0$ is suitable, since the problem $\min\limits_{\lambda}\|a + \lambda a^i\|^2$ has the solution $\lambda_* = -aa^i/\|a^i\|^2 \ge 0$ and, hence,

$$\min_{\lambda}\|a + \lambda a^i\|^2 = \min_{z \in a + \mathrm{aff}\{a^i\}}\|z\|^2 = \min_{\lambda \ge 0}\|a + \lambda a^i\|^2 = \min_{z \in a + \mathrm{Co}\{a^i\}}\|z\|^2.$$

The algorithm with the required modifications is as follows.

Initialization. Set $k = 0$, $z^k = a$, $J_k = \{i_k\}$, and $D_k = \{a^i, i \in J_k\}$, where i_k is such that

$$z^k a^{i_k} = \min_{i = 1, 2, \dots, m} z^k a^i = \alpha_k.$$

If $\alpha_k \ge 0$, then the point $z^{k+1} = a$ (or $x^{k+1} = 0$) solves problem (14) and the algorithm halts. Otherwise, the main iteration loop of the algorithm is executed. Note that $\mathrm{Co}\{D_k\}$ is a suitable cone.

Main iteration loop of the algorithm.

Step 1 (projection onto the subspace). $L_k = a + \mathrm{aff}\{D_k\}$. Solve the problem

$$\min_{z \in a + L_k}\|z\|^2 = \|z^{k+1}\|^2 = \min_{x \in L_k}\|x + a\|^2 = \|x^{k+1} + a\|^2$$

and check whether $\mathrm{Co}\{D_k\}$ is a suitable cone ($x^{k+1} \in \mathrm{Co}\{D_k\}$). Note that $\|z^k\| > \|z^{k+1}\|$ in any case. If $\mathrm{Co}\{D_k\}$ is suitable, then go to Step 2; otherwise, execute Step 3.

Step 2 (verification of optimality). If $z^{k+1}a^i \ge 0$ for all $i = 1, 2 \dots, m$, then z^{k+1} is a solution to problem (14). Indeed, $z^{k+1}u \ge 0$ for any $u \in \mathrm{Co}\{D\}$ and $z^{k+1}(a + u) = \|z^{k+1}\|^2 + z^{k+1}u \ge \|z^{k+1}\|^2$; i.e., the hyperplane $z^{k+1}x = \|z^{k+1}\|^2$ separates z^{k+1} from the set $a + \mathrm{Co}\{D\}$, since $z^{k+1}(x - z^{k+1}) \ge 0$ for $x \in a + \mathrm{Co}\{D\}$. Moreover, for any $x \in a + \mathrm{Co}\{D\}$, we have the estimate

$$\|x\|^2 = \|z^{k+1} + x - z^{k+1}\|^2 = \|z^{k+1}\|^2 + 2z^{k+1}(x - z^{k+1}) + \|z^{k+1} - x\|^2 \ge \|z^{k+1}\|^2.$$

Therefore, z^{k+1} delivers $\min_{z \in a + \mathrm{Co}\{D\}} \|z\|^2$. It is easy to see that the converse is also true: the optimality of z^{k+1} implies $z^{k+1}u \geq 0$ for all $u \in \mathrm{Co}\{D\}$. The algorithm halts.

Otherwise, Step 4 is executed.

Step 3 (inner iteration loop for constructing a new basis). The inner iteration index is set equal to zero ($j = 0$, $w^s = z^{k+1}$, $w^{-1} = z^k$, and $K_s = D_k$).

Inner iteration loop. Determine a maximum $\lambda_s > 0$ such that

$$w^{s+1} = \lambda_s w^s + (1 - \lambda_s) w^{s-1} \in \mathrm{Co}\, K_s.$$

Then, w^{s+1} belongs to a face of $\mathrm{Co}\{K_s\}$ defined by the set of its generatrices $K_{s+1} \subset K_s$, where inclusion is strict. It is easy to show that $\|w^{s+1}\|^2 < \|w^{s-1}\|$. Solve the problem

$$\min_{w \in a + \mathrm{aff}\{K_{s+1}\}} \|w\|^2 = \|w^{s+1}\|^2.$$

If $\mathrm{Co}\{K_{s+1}\}$ is a suitable cone, then set $D_{k+1} = K_{s+1}$ and exit the inner loop. Otherwise, increment the iteration index ($s \longrightarrow s + 1$) and repeat the inner loop. Since K_s is strictly monotone decreasing, the inner loop is executed only a finite number of times.

End of the inner iteration loop.

Step 4. Set $J_{k+1} = J_k$, increment the iteration index ($k \longrightarrow k + 1$), and repeat Step 1.

End of the main iteration loop.

End of the algorithm.

The convergence of this algorithm is proved using its properties pointed out in its formulation (the norm of the current projection decreases monotonically, the number of suitable cones is finite, etc.). The idea behind the proof is similar to that used to prove the convergence of the algorithm in [2] and, for this reason, is omitted.

4. NUMERICAL EXPERIMENTS

The application of the approach described is illustrated by projecting the origin onto high-dimensional polyhedra defined by systems of inequalities with normally distributed random coefficients.

In these tests, the system of inequalities was constructed as a set of m random supporting planes of the n-dimensional sphere of radius $r = \theta\|x^0\|$, $\theta \in (0, 1)$ centered at the random point x^0 with normally distributed coordinates.

To avoid the trivial solution, the set of supporting planes was supplemented by a special one constructed so that it strictly separated the origin from the polyhedron. For this purpose, we used a tangent plane to the sphere passing through the origin. The support vector of this plane can be defined as $\bar{a} = \rho z - \gamma x^0$, where $\rho = \theta\sqrt{1 - \theta^2}$, $\gamma = \theta^2$, and the random vector z is orthogonal to x^0 and belongs to the $(n - 1)$-dimensional unit sphere. The right-hand side of the inequality $\bar{a}x \leq \beta$ was defined as $\beta = \frac{1}{2}\bar{a}x^0 = -\frac{1}{2}\theta^2\|x^0\|^2 < 0$, which guaranteed that the origin was strictly separated from the feasible set. Simultaneously, we required that $\bar{a}x^0 = -\theta^2\|x^0\|^2 < \beta$, which ensured that the feasible set is nonempty.

In a sense, the use of this inequality was an overcautious strategy, since, for large m that considerably exceed n, the randomly generated planes included a sufficient number of planes (about 30–50%) that separated the origin from the feasible set. As a rule, this special plane was not included in the optimal solution, which suggested that its presence in the constraint set did not lead to any specific features.

To facilitate the possibility of comparative experiments, we present the text of the data generator (used to produce the matrix A and the right-hand-side vector b of the system) implemented in the matrix-vector language octave (see [7]). The same generator can be used without modifications in MATLAB:

```
function [ A b ] = genAb(kseed, n, m, r)
randn('seed', kseed);
A       = zeros(m,  n); b       = zeros(m, 1);
v       = randn(n, 1);  v       = v/norm(v);
x0      = randn(n, 1);  sx2     = sumsq(x0);
```

Table

n	m	min	max	ave	std	ntest	fail
1000	1500	470	490	476.7143	7.0170	7	0
1000	1600	478	521	500.3000	14.5911	10	0
1000	1700	502	541	523.1000	11.3964	10	0
1000	1800	516	558	537.3000	14.8702	10	0
1000	1900	549	581	567.5556	11.7698	9	0
1000	2000	568	596	581.2857	10.7659	7	0
1500	1800	535	567	554.0000	12.9228	5	0
1500	1900	567	595	576.7500	12.7639	4	0
1500	2000	574	616	596.7500	17.6517	4	0
1500	2100	595	631	617.6667	19.7315	3	0

```
sx      = norm(x0);
zp      = v - (v'*x0)*x0/sx2;   zp      = zp/norm(zp);
g       = r*sqrt(1-r^2)*norm(x0);
z       = g*zp - r^2*x0;
A(1, :) = z';           b(1)    = -r^3*sumsq(x0)/2;
for i = 2:m
        z = randn(1, n); A(i, :) = z;
        b(i) = z*x0 + r*sx*norm(z);
endfor
endfunction
```

The function genAb has the following parameters: kseed is the initialize of the random-number generator (an integer), n is the dimension of the space of variables, m is the number of inequalities, and r is the relative size of the sphere. The function returns a constraint-set matrix A and a right-hand-side vector b. According to trial runs, for identical values of kseed, this test was identically reproduced in Windows and Linux, but different data were generated in PowerPC.

The numerical results obtained for the projection onto a random polyhedron are presented in the table. Here, n is the dimension of the space of variables; m is the number of inequalities; min, max, and ave are

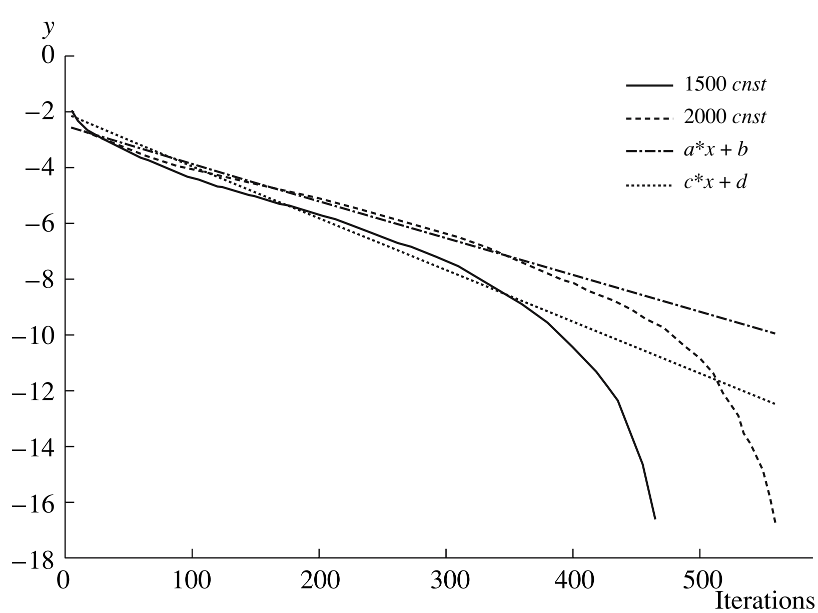

Figure.

the minimum, maximum, and average numbers of iterations in a series of tests, respectively; std is the rms deviation of the number of iterations from the average value; ntest is the number of solved test problems; and fail is the number of problems we failed to solve.

The numerical results suggest that the projection method was fairly stable and the number of iterations increased rather slowly as the number of variables and the number of constraints grew. For the standard updating procedure used for the projector in the modification of the method [2], the complexity of the iteration was $O(n^2)$.

The figure shows the details of the convergence process for problem (14) (projection onto the feasible set) with 1000 variables in the case of 1500 and 2000 constraints. Specifically, the difference between the norm of the projection and the minimum value (on a logarithmic scale) is plotted against the number of iterations. It is seen that the convergence is linear during the major part of computations with acceleration at the beginning and end of the process. The figure also displays linear approximation of convergence, which gives estimates for the ratio of the geometric progression: it is equal to 0.9818 for 1500 constraints (solid line) and to 0.9868 for 2000 constraints (dashed line). The deterioration is only 0.5%, which can be attributed to the uncertainty in the definition of itself. For the dimensions used, the value of the ratio can be regarded as sufficiently small and comparable with theoretical estimates of projective methods.

Because of the large amount of the input data (25–35 Mb), we failed to perform comparative experiments based on free versions of optimizing solvers, such as CPLEX [8] and MINOS. The commercial version of CPLEX (ILOG CPLEX 10.2), which is based on a projective method, produces solutions to similar problems in a considerably smaller number of iterations (10–15), which is typical of interior point methods. However, since the code is not available, it is not possible to determine why the number of iterations is so small. The total computational efficiency cannot be compared either, since the implementation levels are highly different.

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