ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2007, Vol. 47, No. 3, pp. 341–348. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © E.A. Nurminskii, N.B. Shamrai, 2007, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2007, Vol. 47, No. 3, pp. 355–363.

A Method of Local Convex Majorants for Solving Variational-Like Inequalities

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Received March 9, 2006; in final form, September 15, 2006

Abstract—A numerical method based on convex approximations that locally majorize a gap function is proposed for solving a variational-like inequality. The algorithm is theoretically validated and the results of comparison of its numerical efficiency to that of the conventional methods are presented.

DOI: 10.1134/S0965542507030013

Keywords: variational and variational-like inequalities, gap function, convex optimization, method of local convex majorants

INTRODUCTION

At present, many problems in various areas of knowledge (such as mathematical physics, operations research, mathematical economics, etc.) intensively use the apparatus of variational inequalities and their generalizations. The problem of solving a variational inequality denoted by VI(G, X) is to find a point $x \in X$ such that

$$G^{\mathrm{T}}(x)(y-x) \geq 0 \quad \forall y \in X, \tag{1}$$

where $G : X \longrightarrow \mathbb{R}^n$ is a given mapping, and $X \subseteq \mathbb{R}^n$ is a nonempty convex closed set (see [1–3]). The notation used in this paper will be explained below.

The apparatus for solving VI(G, X) is well developed (e.g., see [4, 5] and references therein); however, the applicability conditions for the conventional to generalization (1) are fairly strict: usually, strong monotonicity and the Lipschitz condition for the mappings appearing in the generalized variational inequality or their derivatives are required for convergence (see [6, 7]).

One of the approaches widely used for studying and solving VI(G, X) is the construction of an equivalent optimization problem and the application of mathematical programming methods for its solution. The gap functions of the given inequality play an important part in this transformation (see, e.g., [8]).

This paper deals with a variational-like inequality that is one of the direct generalizations of problem (1) and presents a method based on the constrained optimization of a gap function for its solution. The method is theoretically validated and the numerical efficiency of the proposed approach is compared with that of the conventional approaches. The main idea of the algorithm is to combine the trust region method (see [9]) with the construction of a convex approximating majorant of the gap function. Imposing fairly weak conditions on the mappings defining a variational-like inequality, we can construct a weakly convex gap function (see [10]). From the local optimization viewpoint, constructing convex majorants that are equivalent to such gap functions coincides in essence with calculating the gap function itself. Though the algorithm is local and requires a sufficiently accurate initial approximation, it does not require the mappings to possess monotone-type properties.

1. NOTATION AND DEFINITIONS

This paper uses the following notation: $\mathbb{R}^n$ is an n-dimensional Euclidean space, whose elements are column vectors; T is the transposition symbol; $a_+ = \max\{a, 0\}$ is the positive part of the number a; $\|x\| = \sqrt{x^{\mathrm{T}}x}$

is the Euclidean norm of the vector x; $\nabla F(x)$ is the Jacobi matrix of the mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at the point x (for $m = 1$, it is the gradient of $F(x)$ at x); $U_\delta(\bar{x}) = \{x : \|x - \bar{x}\| \le \delta\}$ is the δ-neighborhood of the point $\bar{x}$; and $f'(x, d) = \lim\limits_{\tau \to +\infty} [f(x + \tau d) - f(x)]/\tau$ is the directional derivative of the function f in the direction d.

In what follows, we will need certain properties of weakly convex functions.

Definition 1 (see [10]). The function $f : X \longrightarrow \mathbb{R}$ is called weakly convex on X if, for any $x \in X$, there exists a nonempty set $f(x)$ of vectors g such that

$$f(y) - f(x) \ge g^{\mathrm{T}}(y - x) + r(x, y) \quad \forall y \in X,$$

where $|r(x, y)|/\|x - y\| \longrightarrow 0$ uniformly with respect to x as $x \longrightarrow y$ in each compact subset of X.

In this paper, the following properties of weakly convex functions are of great interest:

(1) any continuously differentiable function is weakly convex;

(2) if $f(x)$ is a weakly convex function, then it has a directional derivative $f'(x, d)$;

(3) assume that $f(x, y)$ is a weakly convex (with respect to x) function for any fixed $y \in X$ and $Y(x)$ is a set of $y \in Y$ for which $\sup\limits_{y \in Y} f(x, y) = w(x)$; then, $w(x)$ is a weakly convex function and $w'(x, d) = \sup\limits_{y \in Y(x)} f'(x, y, d)$, where $f'(x, y, d)$ is the directional derivative of $f(x, y)$ with respect to x in the direction d.

2. VARIATIONAL-LIKE INEQUALITIES

Let X be a nonempty convex closed subset in $\mathbb{R}^n$ and $G, F : X \longrightarrow \mathbb{R}^m$ be single-valued continuous mappings. The problem of solving a variational-like inequality is denoted by VLI(G, F, X), and it is to find a point $x \in X$ such that

$$G^{\mathrm{T}}(x)[F(y) - F(x)] \ge 0 \quad \forall y \in X. \tag{2}$$

The term *variational-like inequality* was introduced in [11] for the problem of finding a point $x \in X$ such that

$$G^{\mathrm{T}}(x)\eta(y, x) \ge 0 \quad \forall y \in X, \tag{3}$$

where $G : X \longrightarrow \mathbb{R}^n$ and $\eta : X \times X \longrightarrow \mathbb{R}^n$ are given continuous mappings and $X \subseteq \mathbb{R}^n$ is a nonempty convex closed set. Since, for $\eta(y, x) = F(y) - F(x)$ and $m = n$, conditions (2) and (3) coincide, inequality (2) is also a variational-like inequality.

It is shown in [11] that VLI(G, F, X) has a solution if one of the following conditions is fulfilled:

(1) X is a bounded set and, for any fixed $x \in X$, the function $G^{\mathrm{T}}(x)F(y)$ is quasiconvex with respect to $y \in X$;

(2) for any fixed $x \in X$, the function $G^{\mathrm{T}}(x)F(y)$ is convex with respect to $y \in X$ and there exists a compact subset $\Omega \in X$ such that

$$G^{\mathrm{T}}(x)(F(y) - F(x)) < 0 \quad \forall y \in X$$

for any $x \in X \backslash \Omega$. For example, the change of variables in (1) simplifying the feasible domain X yields variational-like inequalities (2). Furthermore, the determination of a generalized solution to a system of inequalities (see [12]) and transport and economic equilibrium problems (see [13–15]) can be written in terms of VLI(G, F, X).

One of the widely used methods of solving problem (1) is based on gap functions. This concept can easily be applied to variational-like inequalities (2).

Definition 2. A gap function for variational inequality (1) (variational-like inequality (2)) is a function $\varphi : X \longrightarrow R \cup \{+\infty\}$ such that

(i) $\varphi(x) \ge 0 \quad \forall x \in X$;

(ii) $x^* \in X$ is a solution to variational inequality (1) (variational-like inequality (2)) if and only if $\varphi(x^*) = 0$ and $x^* \in X$.

It is clear that, in this case, VLI(G, F, X) is equivalent to the constrained optimization problem

$$\min_{x \in X} \varphi(x). \tag{4}$$

Variational inequality (1) is the subject of many works devoted to various types of gap functions. The most general form of a gap function is proposed in [16]:

$$\nu(x) = \sup_{y \in X} \{ f(x) - f(y) + [G(x) - \nabla f(x)]^{\mathrm{T}}(x - y) \}, \tag{5}$$

where $f: \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function differentiable on X. Note that (5) is a convex programming problem; however, it is difficult to guarantee that $\nu(x)$ is convex and differentiable.

In [16], along with primal gap function (5), its dual formulation is considered:

$$\tilde{\nu}(y) = \inf_{x \in X} \{ f(x) - f(y) + [G(x) - \nabla f(x)]^{\mathrm{T}}(x - y) \}. \tag{6}$$

Therefore, the equivalent optimization problem for (1) is to maximize $\tilde{\nu}(y)$ with respect to $y \in X$. In general, (6) is not a convex programming problem; however, $\tilde{\nu}(y)$ is concave.

Particular forms of functions (5) and (6) with $f \equiv 0$ were proposed in [17] for finding equilibriums in the problems of game theory and were studied later in [18] and other works. The main results were obtained for the case when the supremum in (5) is attainable at a unique point and, consequently, the function $\nu(x)$ is differentiable. Another way to make $\nu(x)$ differentiable is to use strongly convex functions $f(x)$, for example, $f(x) = (1/2)\|x\|^2$ (see [19]). Differentiable gap functions were also studied in [8, 20] and other works.

In this paper, the following gap function is considered for VLI(G, F, X):

$$\varphi(x) = \sup_{y \in X} G^{\mathrm{T}}(x)[F(x) - F(y)]. \tag{7}$$

It is easy to see that this function satisfies conditions (i) and (ii) in Definition 2.

Indeed, for any $x \in X$, it holds that

$$\varphi(x) = \sup_{y \in X} G^{\mathrm{T}}(x)[F(x) - F(y)] \ge G^{\mathrm{T}}(x)[F(x) - F(x)] = 0. \tag{8}$$

If $x^* \in X$ is a solution to VLI(G, F, X), then

$$G^{\mathrm{T}}(x^*)[F(x^*) - F(y)] \le 0 \quad \forall y \in X$$

and, consequently,

$$\varphi(x^*) = \sup_{y \in X} G^{\mathrm{T}}(x^*)[F(x^*) - F(y)] \le 0.$$

However, (8) implies that $\varphi(x^*) \ge 0$; consequently, $\varphi(x^*) = 0$.

On the contrary, assume that, for a certain $x^* \in X$, it holds that

$$0 = \varphi(x^*) = \sup_{y \in X} G^{\mathrm{T}}(x^*)[F(x^*) - F(y)] \ge G^{\mathrm{T}}(x)[F(x^*) - F(y)] \quad \forall y \in X,$$

consequently, x^* is a solution to the problem VLI(G, F, X).

Being conceptually simple, function (7) has a number of disadvantages: it is generally not a differentiable function and, moreover, it is difficult to ensure its convexity for nonlinear G and F. However, if G and F are continuously differentiable and the supremum in (7) is attainable, then $\varphi(x)$ is a weakly convex function (see [10]).

In this paper, to solve problem (4), we propose to construct a convex approximation of gap function (7), which is almost equivalent to $\varphi(x)$ from the viewpoint of its optimization in the neighborhood of an approximate solution.

3. THE LOCAL CONVEX MAJORANT OF THE GAP FUNCTION

Below, we assume that the mappings G and F are twice continuously differentiable on X and the supremum in (7) is attainable for any $x \in X$.

Fix a point $\bar{x} \in X$ and consider its δ-neighborhood $U_\delta(\bar{x})$, where $\delta > 0$ is sufficiently small. The neighborhood $U_\delta(\bar{x})$ is chosen so that, for a certain $R > 0$ and any $y \in X$, $x_\delta \in U_\delta(\bar{x})$, and $z \in \mathbb{R}^n$, it holds that

$$\left| z^{\mathrm{T}} H_1(x_\delta) z \right| \le R\|z\|^2, \quad \left| z^{\mathrm{T}} H_2(x_\delta, y) z \right| \le R\|z\|^2,$$

where $H_1(x_\delta)$ and $H_2(x_\delta, y)$ are the matrices of the second derivatives of the functions $G^{\mathrm{T}}(x)F(x)$ and $-G^{\mathrm{T}}(x)F(y)$, respectively, taken at the point x_δ.

Introduce the notation

$$
\begin{aligned}
h(x, y) &= G^{\mathrm{T}}(x)[F(x) - F(y)],\\
c_0(\bar{x}) &= G^{\mathrm{T}}(\bar{x})F(\bar{x}),\\
C(\bar{x}) &= F^{\mathrm{T}}(\bar{x})\nabla G(\bar{x}) + G^{\mathrm{T}}(\bar{x})\nabla F(\bar{x}),\\
A(\bar{x}, z) &= \nabla G(\bar{x})z + G(\bar{x}),
\end{aligned}
\tag{9}
$$

and assume that $Z(x) = \{z : \|z\| \le \delta, x + z \in X\}$ is the set of feasible displacements from the point x whose norms do not exceed δ.

Let $z \in Z(\bar{x})$ and $x = \bar{x} + z$. The quantity $h(x, y)$ satisfies the bound

$$
\begin{aligned}
h(x, y) &= h(\bar{x} + z, y) = G^{\mathrm{T}}(\bar{x} + z)[F(\bar{x} + z) - F(y)] = c_0(\bar{x}) + C(\bar{x})z - F^{\mathrm{T}}(y)A(\bar{x}, z)\\
&+ (1/2)z^{\mathrm{T}}H_1(x_\delta)z + (1/2)z^{\mathrm{T}}H_2(x_\delta)z \le c_0(\bar{x}) + C(\bar{x})z + R\|z\|^2 - F^{\mathrm{T}}(y)A(\bar{x}, z) = \tilde{h}(\bar{x}, y, z).
\end{aligned}
\tag{10}
$$

Consider the function

$$
\psi(\bar{x}, z) = \sup_{y \in X}\tilde{h}(\bar{x}, y, z) = c_0(\bar{x}) + C(\bar{x})z + R\|z\|^2 - \inf_{y \in X} F^{\mathrm{T}}(y)A(\bar{x}, z), \tag{11}
$$

where the supremum is assumed to be attainable for all $\bar{x} \in X$ and $z \in Z(\bar{x})$. Note that $\psi(\bar{x}, z)$ is convex with respect to z.

Bound (10) implies that

$$
\sup_{y \in X} h(\bar{x} + z, y) = \varphi(\bar{x} + z) \le \psi(\bar{x}, z),
$$

and, for $z = 0$, the inequality turns into an equality.

The following theorem establishes a relationship between variational-like inequality (2) and the function $\psi(\bar{x}, z)$.

Theorem 1. *The point x^* is a solution to the problem* VLI(G, F, X) *if and only if $z = 0$ is a solution to the problem*

$$
\min_{z \in Z(x^*)} \psi(x^*, z) \tag{12}
$$

and $\psi(x^*, 0) = 0$.

Proof. Assume that x^* is a solution to the problem VLI(G, F, X) and, consequently, x^* is a solution to problem (4) and $\varphi(x^*) = 0$. Then, for any $z \in Z(x^*)$, we have

$$
\psi(x^*, 0) = \varphi(x^*) \le \varphi(x) = \varphi(x^* + z) \le \psi(x^*, z).
$$

Thus, $z = 0$ is a solution to problem (12) and $\psi(x^*, 0) = 0$.

Conversely, assume that $z = 0$ is a solution to problem (12) and $\psi(x^*, 0) = 0$. Since $x^* \in X$ and it holds that

$$
0 = \psi(x^*, 0) = \varphi(x^*),
$$

we have by property (ii) in Definition 2 that the point x^* is a solution to VLI(G, F, X).

Denote by $\varphi'(x, d)$ and $\psi'(\bar{x}, z, d)$ the derivatives in the directions of functions (7) and (11) with respect to the variables x and z, respectively; that is,

$$
\varphi'(x, d) = \lim_{\tau \to +0}[\varphi(x + \tau d) - \varphi(x)]/\tau, \tag{13}
$$

$$
\psi'(\bar{x}, z, d) = \lim_{\tau \to +0}[\psi(\bar{x}, z + \tau d) - \psi(\bar{x}, z)]/\tau. \tag{14}
$$

Limits (13) and (14) exist by the assumptions that the corresponding suprema in (7) and (11) are attainable, $\varphi(\cdot)$ is weakly convex, and $\psi(\bar{x}, \cdot)$ is convex (see [10]).

By the following lemma, $\psi(\bar{x}, z)$ is a good approximation of $\varphi(x)$ in a neighborhood of $\bar{x}$.

Lemma. *Assume that* $\bar{x} \in X$. *Then, for any* $d \in Z(\bar{x})$, *it holds that*

$$\varphi'(\bar{x}, d) = \psi'(\bar{x}, 0, d).$$

Proof. Assume that $Y(x)$ is the set of $y \in X$ for which the supremum in (7) is attained. Since $\varphi(x)$ is weakly convex, we have

$$\varphi'(x, d) = \sup_{y \in Y(x)} h'(x, y, d) = \sup_{y \in Y(x)} h'_x(x, y)d = C(x)d - \inf_{y \in Y(x)} [F^{\mathrm{T}}(y)\nabla G(x)d].$$

Assume that $\tilde{Y}(\bar{x}, z)$ is the set of $y \in X$ for which the supremum in (11) is attained. Similarly, due to the weak convexity of $\psi(\bar{x}, z)$ with respect to z, we have

$$\psi'(\bar{x}, z, d) = \sup_{y \in Y(\bar{x}, z)} \tilde{h}'(\bar{x}, y, z, d) = \sup_{y \in Y(\bar{x}, z)} \tilde{h}'_z(\bar{x}, y, z)d = C(\bar{x})d + 2Rz^{\mathrm{T}}d - \inf_{y \in \tilde{Y}(\bar{x}, z)} [F^{\mathrm{T}}(y)\nabla G(x)d].$$

When $x = \bar{x}$ and $z = 0$, the sets $Y(x)$ and $\tilde{Y}(\bar{x}, z)$ coincide; that is, $Y(\bar{x}) = \tilde{Y}(\bar{x}, 0)$. Consequently,

$$\varphi'(\bar{x}, d) = \psi'(\bar{x}, 0, d).$$

This lemma implies, in particular, that $\bar{x}$ is a stationary point for $\varphi(x)$ if and only if $z = 0$ is a stationary point of the function $\psi(\bar{x}, z)$.

4. THE METHOD OF LOCAL CONVEX MAJORANTS

Using the fact that $\varphi(x)$ ($x \in U_\delta(\bar{x})$) can be locally approximated by the convex majorant $\psi(\bar{x}, z)$ ($z \in Z(\bar{x})$) described by the lemma, we propose the following algorithm for solving variational-like inequality (2).

Algorithm Initialization

Choose a point $x^0 \in X$ and consider the set $X^0 = \{x : \varphi(x) \le \varphi(x^0)\}$. Determine $\delta > 0$ such that, for any $\bar{x} \in X^0$ and $z \in Z(\bar{x})$, it holds that

$$\varphi(\bar{x} + z) \le \psi(\bar{x}, z).$$

Take the initial point $\bar{x}^0 \in X^0$. Set $k = 0$.

Algorithm Iteration

Step 1. Solve the problem

$$\min_{z \in Z(\bar{x}^k)} \psi(\bar{x}^k, z) = \psi(\bar{x}^k, z^k). \tag{15}$$

Step 2. If $z^k = 0$ and $\psi(\bar{x}^k, 0) = 0$, then $\bar{x}^k$ is a solution to VLI(G, F, X). If $z^k = 0$ but $\psi(\bar{x}^k, 0) \ne 0$, then either VLI(G, F, X) has no solutions or we have found a local minimum of the gap function φ. The algorithm stops.

Step 3. Set $\bar{x}^{k+1} = \bar{x}^k + z^k$ and $k = k + 1$ and go to Step 1.

The convergence of this algorithm is established by the following theorem.

Theorem 2. *Assume that the vector* x^0 *is such that the set* X^0 *does not contain the local minima and stationary points of the gap function* $\varphi(x)$ *that differ from the global ones. Then, the sequence* $\{\bar{x}^k\}$ *generated by the algorithm converges to a solution to* VLI(G, F, X).

Proof. Let z^k be a solution to problem (15). Then,

$$\varphi(\bar{x}^k) = \psi(\bar{x}^k, 0) \ge \psi(\bar{x}^k, z^k) \ge \varphi(\bar{x}^k + z^k) = \varphi(\bar{x}^{k+1}).$$

Thus, the sequence $\{\varphi(\bar{x}^k)\}$ monotonically decreases and there exists a limit $\varphi^* = \lim_{k \to \infty} \varphi(\bar{x}^k)$.

Fix a certain $z \in Z(\bar{x}^k)$. By construction,

$$\psi(\bar{x}^k, z^k) \le \min_{\theta \in [0,1]} \psi(\bar{x}^k, \theta z) \le \psi(\bar{x}^k, 0) + \min_{\theta \in [0,1]} [\theta\psi'(\bar{x}^k, 0, z) + o(\theta)].$$

Suppose that, for an arbitrarily large K, there exist $k > K$, $\gamma > 0$, and $\bar{z}^k \in Z(\bar{x}^k)$ such that $\psi(\bar{x}^k, 0, \bar{z}^k) \le -\gamma < 0$; then,

$$\psi(\bar{x}^k, z^k) \le \psi(\bar{x}^k, 0) + \theta[-\gamma + o(\theta)/\theta] \le \psi(\bar{x}^k, 0) - \theta\gamma/2$$

for sufficiently small $\theta > 0$. Fixing such a θ, we obtain

$$\varphi(\bar{x}^{k+1}) \le \psi(\bar{x}^k, 0, z^k) \le \psi(\bar{x}^k, 0) - \varepsilon = \varphi(\bar{x}^k) - \varepsilon, \tag{16}$$

where $\varepsilon = \theta\gamma > 0$. Passing to the limit in (16) as $K \longrightarrow \infty$, we obtain

$$\varphi^* \le \varphi^* - \varepsilon,$$

which is impossible. In particular, this implies that

$$\lim_{k \to \infty} \inf_{z \in Z(\bar{x}^k)} \psi'(\bar{x}^k, 0, z) = \lim_{k \to \infty} \inf_{z \in Z(\bar{x}^k)} \varphi'(\bar{x}^k, z) \ge 0,$$

and, since $Z(x)$ is lower semicontinuous, we obtain the inequality

$$\varphi'(\bar{x}, z) \ge \inf_{z \in Z(\bar{x})} \varphi'(\bar{x}, z) \ge \lim_{k \to \infty} \inf_{z \in Z(\bar{x}^k)} \varphi'(x^k, z) \ge 0$$

for any $z \in Z(\bar{x})$ and any limit point $\bar{x}$ of the sequence $\{\bar{x}^k\}$. Thus, at $\bar{x}$, the necessary extremum conditions are satisfied for $\varphi(x)$ $(x \in X)$ and, consequently, $\bar{x}$ is a solution to problem (2).

5. A NUMERICAL EXAMPLE

To demonstrate the advantages of the representation of the original problem in the form of a variational-like inequality and of the application of the method of local convex majorants, we compare the proposed approach with the conventional one based on minimizing a regularized gap function. Consider the examples of variational and variational-like inequalities generated by the optimization problem

$$\min_{u \in U} g(u), \tag{17}$$

where $g(u) = (1/4)(u_1 - u_2)^2 - (1/2)(u_1 + u_2)$ and $U = \{u : \|u\|^2 \le 1, u \ge 0\}$, whose solution is the point $(\sqrt{2}/2, \sqrt{2}/2)$.

The Regularization Method (RM)

Problem (17) is equivalent to the variational inequality

$$Q^{\mathrm{T}}(u)(v - u) \ge 0 \quad \forall v \in U, \tag{18}$$

where $Q^{\mathrm{T}}(u) = (1/2)(u_1 - u_2 - 1, u_2 - u_1 - 1) = \nabla g(u)$.

Set $f(u) = (1/2)\|u\|^2$ in (5). In this case, a regularized gap function for (18) has the form

$$\nu(u) = \|u - a(u)\|^2 - [\|a(u)\| - 1]_+^2, \tag{19}$$

where $a^{\mathrm{T}}(u) = (1/2)(1 + u_1 + u_2, 1 + u_1 + u_2)$. This regularization is most widely used, and $\nu(u)$ is the Fukushima gap function (see [19]). Note that $\nu(u)$ is nonconvex in this case.

To minimize $\nu(u)$ for $u \in U$, we used the method of modified Lagrangian functions implemented in the MINOS software package (see [21). The decrease of the values of $\nu(u)$ as the algorithm operates is shown in the figure by the dashed line MR.

One can suppose that it is the discontinuity of the second derivatives of gap function (19) that does not allow attaining a more than linear rate of convergence for the most part of the optimization process. The computational speedup at the end of the computations occurs because $\nu(u)$ has continuous second derivatives in a small neighborhood of a solution where the calculations are performed.

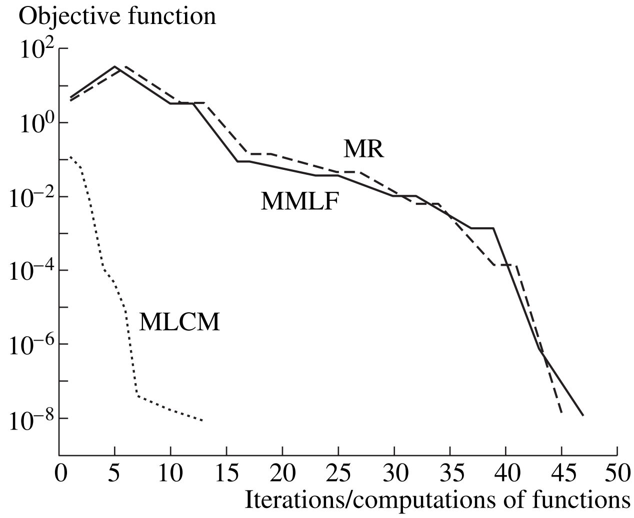

Figure.

For comparison, original problem (17) was solved by the method of modified Lagrangian functions. The decrease of the objective function $g(u)$ is also shown in the figure by the solid line MMLF. It is clear that both RM and MMLF approaches to solving (17) have similar computational efficiencies though the problems under study have significantly different properties. This seems to be due to the fact that, in both formulations, the constraints imposed on U are nonlinear, which presents the main computational difficulty.

The Method of Local Convex Majorants (MLCM)

Since the method proposed in Section 4 can be applied to a more general class of variational inequalities than (18), the feasible domain U can be simplified. The change of variables $u_1^2 = x_1$, and $u_2^2 = x_2$ reduces inequality (18) to variational-like inequality (2) with the main mappings

$$G^{\mathrm{T}}(x) = (1/2)(\sqrt{x_1}-\sqrt{x_2}-1, \sqrt{x_2}-\sqrt{x_1}-1), \quad F^{\mathrm{T}}(x) = (\sqrt{x_1}, \sqrt{x_2}), \tag{20}$$

and the feasible domain

$$X = \{x : x_1 + x_2 \le 1, x_1 \ge 0, x_2 \ge 0\}. \tag{21}$$

Note that X is now described only by linear constraints (21).

Gap function (7) for (2) subject to (20) and (21) has the form

$$\varphi(x) = G^{\mathrm{T}}(x)F(x) + \begin{cases} \|G(x)\|, & \text{if } G(x) > 0, \\ \mu(x), & \text{otherwise,} \end{cases} \tag{22}$$

where $\mu(x) = (1/2)\max\{0, 1-\sqrt{x_1}+\sqrt{x_2}, 1+\sqrt{x_1}-\sqrt{x_2}\}$.

For $\bar{x} \in X$ and $\delta > 0$, a locally convex majorant has the form

$$\psi(\bar{x}, z) = c_0(\bar{x}) + C(\bar{x})z + R\|x\|^2 + \begin{cases} \|A(\bar{x}, z)\|, & \text{if } A(\bar{x}, z) < 0, \\ \mu(\bar{x}, z), & \text{otherwise,} \end{cases}$$

where $z \in Z(\bar{x})$, $\mu(\bar{x}, z) = (1/2)\max\{0, a_1(\bar{x}, z), a_2(\bar{x}, z)\}$, $c_0(\bar{x})$, $C(\bar{x})$, and $A(\bar{x}, z)$ are defined in (9), while $a_1(\bar{x}, z)$ and $a_2(\bar{x}, z)$ are the components of the vector function $A(\bar{x}, z)$ for variational-like inequality (2) for (20) and (21).

The point $x^0 = [0.2, 0.4]^{\mathrm{T}}$ was taken as an initial point and the parameters of the algorithm were $\delta = 0.1$ and $R = 0.5$. The algorithm was implemented using the Matlab-like language of Octave (see [22]). The cutting plane method (see [23]) was applied to solve convex optimization problem (15).

In the figure, the dotted line MLCM shows the decrease of the gap function $\varphi(x)$ as a function of the algorithm iterations. The results of numerical experiments demonstrate a much quicker convergence of the MLCM in comparison with the RM and MMLF. It should be noted that the iterations in the MLCM are a sufficiently complex procedure because optimization problem (15) must be solved at Step 1; however, the efficiency of its solution can be significantly improved if the algorithm starts at a previous optimum or other hot-start methods are used. Under these conditions, the total computational efficiency of the MLCM is expected to be sufficiently high.

This work was supported by the "Leading Scientific Schools" Program (project no. NSh-9004.2006.1) and by the Far East Division of the Russian Academy of Sciences (project no. 01-052).

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