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The Use of Additional Diminishing Disturbances in Fejer Models of Iterative Algorithms

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Abstract—The behavior of Fejer processes with diminishing disturbances generated by a small shift in the argument of the Fejer operator is studied. It is shown that, if the operator is locally strongly Fejer, a diminishing disturbance does not prevent convergence to an attracting set. At the same time, such a disturbance can be used to furnish the process with additional properties that ensure convergence to certain subsets of the attracting set. In particular, based on this scheme, new parallel decomposition methods for optimization problems can be suggested that do not require that the constraints possess a specific structure.

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INTRODUCTION

Since the publication of Eremin's fundamental works [1–3], the concept of Fejer processes has been successfully used in numerous studies on the convergence of iterative processes designed for solving systems of equalities/inequalities, optimization problems, etc. The present-day theory of Fejer processes can be found in [4, 5]. In this paper, we analyze the behavior of Fejer processes with additional diminishing disturbances, i.e., recurrence sequences generated by relations of the form $x^{s+1} = F(x^s + z^s)$, $s = 0, 1, \ldots$, where F belongs to a subclass of Fejer operators and z^s is an additional diminishing disturbance ($z^s \longrightarrow 0$ as $s \longrightarrow \infty$). It is assumed that z^s is determined by the current state of the process x^s. On the one hand, z^s can be viewed as a noise. On the other hand, due to this disturbance, we can furnish the Fejer process with some desirable properties, thus achieving an additional positive effect.

1. NOTATION AND DEFINITIONS

The consideration below is associated primarily with the finite-dimensional Euclidean space E equipped with the inner product xy and the norm $\|x\| = \sqrt{xx}$. Situations where a vector x is multiplied by a scalar factor α are usually clear from the context. The real line is denoted by $\mathbb{R}$.

In what follows, we use the general convergence conditions for iterative processes [9], which are, in a sense, a discrete version of the Lyapunov asymptotic stability conditions. According to these conditions, the sequence $\{x^s\}$ generated by an iterative process has a limit point in a given set X_* if the following conditions are satisfied.

Condition 1. The sequence $\{x^s\}$ is bounded.

Condition 2. For each subsequence $\{x^{n_k}\}$ converging to $x' \notin X_*$, there exists $\varepsilon > 0$ such that, for every n_k, there is an index $m_k > n_k$ such that $\|x^{n_k} - x^s\| \le \varepsilon$ for $n_k \le s < m_k$ and $\|x^{n_k} - x^{m_k}\| > \varepsilon$.

Condition 3. There exists a continuous function $W(x)$ such that, for any subsequence $\{x^{n_k}\}$ converging to $x' \notin X_*$ and for the corresponding subsequence $\{x^{n_k}\}$ (which exists by Condition 2), there is a subsequence $\{p_k\}$ with $n_k < p_k \le m_k$ such that

$$\limsup_{k\to\infty} W(x^{p_k}) < \liminf_{k\to\infty} W(x^{n_k}) = W(x').$$

For each limit point of $\{x^s\}$ to be in X_*, it is sufficient that the following two additional conditions hold.

Condition 4. If $x^{n_k} \longrightarrow x^* \in X_*$, then $\|x^{n_k+1} - x^{n_k}\| \longrightarrow 0$.

Condition 5. The set $W(X_*) = \{W(x^*) : x^* \in X_*\}$ is such that $\mathbb{R}\backslash W(X_*)$ is everywhere dense.

Meaningfully, Condition 2 prevents the iterative process $\{x^s\}$ from being stuck at points that are not in X_*. Condition 3 implicitly forbids limit cycles that do not pass through points of X_* and is an analogue of the condition that the total derivative of the Lyapunov function along the trajectory of the dynamic process is negative. Condition 4 prevents the points of $\{x^s\}$ from jumping away from the limit set X_*. Condition 5, in conjunction with Conditions 1–4, guarantees the convergence of $\{W(x^s)\}$. Together with Condition 3, this prevents $\{x^s\}$ from having limit points that are not in X_*.

2. CONVERGENCE OF FEJER SEQUENCES WITH DIMINISHING DISTURBANCES

For the goals of this paper, the Fejer operator is defined with respect to a given set V as follows.

Definition 1. An operator $F : E \longrightarrow E$ is called Fejer (with respect to a given set V) if $F(v) = v$ for $v \in V$ and $x \in E$

$$\|F(x) - v\| \le \|x - v\| \tag{1}$$

for all $v \in V$.

The set V is usually clear from the context and is hereafter assumed to be closed and (primarily for simplicity) bounded. In addition to the definition, we also assume that $F(x)$ is continuous on an open extension of V.; i.e., it is assumed that there exists an open $\tilde{V}$ such that $V \subset \tilde{V}$ and $F(x)$ is continuous on $\tilde{V}$. Since $F(x)$ is continuous on V by definition, we actually mean that $F(x)$ is continuous on the boundary of V.

Given a Fejer operator F and an initial point x^0, we can construct an iterative Fejer process $x^{s+1} = F(x^s)$ ($s = 0, 1, \ldots$) that models a computational algorithm for determining a point or points of V. Property (1) or rather its various stronger versions guarantee that the elements of $\{x^s\}$ converge to V in a certain sense.

Bearing in mind the subsequent applications, we propose the following properties of F, which are also stronger than (1).

Definition 2. A Fejer operator F is called *locally strongly Fejer* if, for any $\bar{x} \notin V$, there exists a neighborhood U of zero and a number $\alpha \in [0, 1)$ such that

$$\|F(x) - v\| \le \alpha\|x - v\| \tag{2}$$

for all $v \in V$ and $x \in \bar{x} + U$.

Theorem 1. *Let V be a closed and bounded set; F be a locally strongly Fejer operator; the sequence $\{x^s\}$ generated by the recurrence relations*

$$x^{s+1} = F(x^s + z^s), \quad s = 0, 1, \ldots, \tag{3}$$

be bounded; and $z^s \longrightarrow 0$ as $s \longrightarrow \infty$. Then, for arbitrary x^0, all the limit points of $\{x^s\}$ belong to V.

Proof. The convergence is proved by verifying Conditions 1–5 with the set $X_* = V$ and the Lyapunov function $W(x) = \inf_{v \in V}\|x - v\| = \rho(x, V)$. Since Condition 1 holds by the theorem assumption, we begin the verification from Condition 2.

Assume that there exists a subsequence x^{n_k} converging to $x' \notin V$ with $\rho(x', V) \ge 2\delta > 0$. A neighborhood U is chosen so that, for $x \in x' + 4U$, condition (2) is satisfied with some $\alpha < 1$ and $\rho(x, V) \ge \delta > 0$. If we assume that for all $s > n_k x^s \in x^{n_k} + U \subset x' + 2U$, then $x^s + z^s \in x' + 4U$ for sufficiently large k and $x^{s+1} = F(x^s + z^s)$ satisfies the relation

$$\|x^{s+1} - v\| = \|F(x^s + z^s) - v\| \le \alpha\|x^s + z^s - v\| \le \alpha\|x^s - v\| + q\|z^s\| \le (1+\alpha)/2\|x^s - v\| = \beta\|x^s - v\|$$

for all $v \in V$, where $\beta < 1$, whenever $\|z^s\| \leq (1 - \alpha)/\alpha\|x^s - v\| = \gamma_s$. The right-hand side of the last inequality is bounded below by a positive number: $\gamma_s \geq \rho(x^s, V)(1 - q)/q \geq \delta(1 - \alpha)/\alpha > 0$. Therefore, this inequality holds for sufficiently large k.

Iterating the resulting inequality $\|x^{s+1} - v\| \leq \beta\|x^s - v\|$ over $s = n_k, n_k + 1, \dots, t - 1$, we obtain $\|x^t - v\| \leq \beta^{t - n_k}\|x^{n_k} - v\| \longrightarrow 0$ as $t \longrightarrow \infty$, which contradicts $\|x^t - v\| \geq \rho(x^t, V) \geq \delta > 0$ since, by assumption, $x^t \in x^{n_k} + U \subset x' + 2U$ for all $t > n_k$. This contradiction proves that, for any k, there exists m_k such that

$$\|x^t - x^{n_k}\| \leq \varepsilon, \quad n_k \leq t < m_k \quad \text{and} \quad \|x^{m_k} - x^{n_k}\| > \varepsilon > 0$$

for some $\varepsilon > 0$ such that $\|z\| \leq \varepsilon$ implies $z \in U$. This proves Condition 2.

Furthermore,

$$\rho(x^{m_k}, V) \leq \beta^{m_k - n_k}\rho(x^{n_k}, V) \leq (\beta\rho(x^{n_k}, V) = \rho(x^{n_k}, V) - (1 - \beta)\rho(x^{n_k}, V)) \leq \rho(x^{n_k}, V) - \theta$$

for $\theta = (1 - \beta)\rho(x^{n_k}, V) \geq (1 - \beta)\delta > 0$. Passage to the limit gives

$$\limsup_{k \to \infty}\rho(x^{m_k}, V) \leq \liminf_{k \to \infty}\rho(x^{n_k}, V) - \theta < \liminf_{k \to \infty}\rho(x^{n_k}, V);$$

i.e., Condition 3 holds with $p_k = m_k$.

It is easy to see that Condition 4 also holds. Indeed, let $x^{n_k} \longrightarrow \bar{x} \in V$. Denote by v^s the solution to the problem $\min_{v \in V}\|x^s - v\|$. By construction, $\|x^{n_k} - v^{n_k}\| \longrightarrow 0$ as $k \longrightarrow \infty$ and, since V is bounded, $\|F(x^{n_k} - F(v^{n_k}))\| \longrightarrow 0$. Moreover,

$$\|x^{n_k+1} - x^{n_k}\| = \|F(x^{n_k}) - x^{n_k}\| = \|F(x^{n_k}) - F(v^{n_k} + v^{n_k} - x^{n_k})\| \leq \|F(x^{n_k}) - F(v^{n_k})\| + \|v^{n_k} - x^{n_k}\| \longrightarrow 0$$

as $k \longrightarrow \infty$, which proves Condition 4. Condition 5 is obviously fulfilled, since $\rho(v, V) = 0$ for all $v \in V$. The proof is complete.

These results can be extended to nonstationary Fejer sequences of the form

$$x^{s+1} = F_s(x^s + z^s), \quad s = 0, 1, \dots, \tag{4}$$

where F_s is chosen from a finite family of Fejer operators.

Corollary 1. $\mathcal{F} = \{P_1, \dots, P_m\}$ is a family of operators P_i such that, for any $x \notin V$, there exists P_i that is locally strongly Fejer at x; $z^s \longrightarrow 0$ as $s \longrightarrow \infty$; and $F_s = P_{i_s}$, where P_{i_s} is a locally strongly Fejer operator at x^s. Then, if the sequence $\{x^s\}$ is bounded, all its limit points belong to V.

Proof. It repeats word for word the proof of Theorem 1 with the only difference being that the constant $\alpha < 1$ of the strong Fejer property for the operators F_s satisfies an estimate of the form

$$\|F_s(x) - v\| = \|P_{i_s}(x) - v\| \leq \max_{i \in I(x)}\alpha_i\|x - v\| \leq \max_{z \in x' + 4U}\max_{i \in I(z)}\alpha_i\|x - v\| = \alpha\|x - v\|,$$

where $I(x)$ is the set of i such that P_i is locally strongly Fejer at x with a constant $\alpha_i < 1$, $i_s \in I(x)$. Here, $x \in x' + 4U$, where x' is a limit point of $\{x^s\}$ chosen to verify Conditions 2 and 3 and U is a sufficiently small neighborhood of zero.

Theorem 1 shows that, under rather mild conditions, the diminishing disturbance does not prevent convergence to some set of fixed points, which is the main result in the theory of Fejer processes. Below, we show how diminishing disturbances can be used to make processes (3) and (4) convergent to certain subsets of V.

We introduce the concept of a bounded attractant as a vector field that is directed inside V toward a subset of V.

Definition 3. A point-to-set mapping $\Phi : V \longrightarrow E$ is called a bounded attractant of $Z \subset V$ if $g(z - x) \geq 0$ for all $x \in V \setminus Z$, $g \in \Phi(x)$, and $z \in Z$.

In fact, a somewhat stronger property is necessary to substantiate the desired convergence.

Definition 4. An attractant Φ is called a strong bounded attractant of Z if, for any $x' \in V\setminus Z$, there exists a neighborhood U of zero such that

$$g(z-x) \geq \delta > 0$$

for all $z \in Z$, $x \in x' + U$, and $g \in \Phi(x)$ and for some $\delta > 0$.

Theorem 2. *Let F be a locally strongly Fejer operator; Φ be a bounded strong attractant of $Z \subset V$ that is upper semicontinuous on some open set $\tilde{V} \supset V$; and the sequence $\{x^s\}$ be generated by*

$$x^{s+1} = F(x^s + \lambda_s \Phi(x^s)), \tag{5}$$

where x^0 is an arbitrary initial state, $\lambda_s \longrightarrow +0$, and $\Sigma\lambda_s = \infty$. *Then, if* $\{x^s\}$ is bounded, any of its limit points belongs to Z.

Proof. We use Conditions 1–5 with the Lyapunov function $W(x) = \inf_{v \in Z} \|x - v\|$.

First, we verify Condition 2. Let $x' \notin Z$ be a limit point of $\{x^s\}$. By Theorem 1, at least $x' \in V$. By Definition 4, there exists a neighborhood U of zero such that $g(z-x) \geq \delta > 0$ for all $z \in Z$ and $x \in x' + 4U$. Since $\Phi(\cdot)$ is upper semicontinuous, we may also assume that $\|g\| \leq C < \infty$ for all $g \in \Phi(x)$ and $x \in x' + 4U$. If $\{n^{n_k}\} \longrightarrow x'$ as $k \longrightarrow \infty$, then $x^{n_k} \in x' + U$ for sufficiently large k. If $x^s \in x^{n_k} + U$ for all $s > n_k$, then $x^s \in x' + 2U$ and $x^s + z^s \in x' + 4U$, since z^s is small. Then, for $v \in Z$ and $\bar{x}^{s+1} = x^s + \lambda_s g^s$, where $g^s = \Phi(x^s)$, we have

$$\|x^{s+1} - z\|^2 \leq \|x^s + \lambda_s g^s - z\|^2 = \|x^s - z\|^2 + \lambda_s^2 \|g^s\|^2 - 2\lambda_s g^s(z - x^s) \leq \|x^s - z\|^2 - \theta\lambda_s$$

for sufficiently large s and some $\theta > 0$. Therefore, $\|\bar{x}^{s+1} - z\| \leq \|x^s - z\| - \gamma\lambda_s$ for some $\gamma > 0$. The local strong Fejer property implies that

$$\|x^{s+1} - z\| = \|F(\bar{x}^{s+1}) - z\| \leq q\|\bar{x}^{s+1} - z\| \leq q\|x^s - z\| - q\gamma\lambda_s \leq \|x^s - z\| - \gamma'\lambda_s,$$

where $\gamma' > 0$. Calculating the infimums gives

$$W(x^{s+1}) \leq W(x^s) - \gamma'\lambda_s, \tag{6}$$

and summing (6) over s yields $W(x^s) \longrightarrow -\infty$, which is not possible.

Therefore, there exists $m_k > n_k$ such that, for some $\varepsilon > 0$ such that $\|u\| \leq \varepsilon$ implies $u \in U$, we have

$$\|x^{m_k} - x^{n_k}\| > \varepsilon, \quad \|x^s - x^{n_k}\| \leq \varepsilon, \quad n_k \leq s < m_k,$$

which proves Condition 2. Relation (6) holds for $n_k \leq s < m_k$. Therefore, summing gives

$$W(x^{m_k}) \leq W(x^{n_k}) - \gamma' \sum_{s = n_k}^{m_k - 1} \lambda_s.$$

The sum $\sum_{s = n_k}^{m_k - 1} \lambda_s$ is easily estimated from below:

$$\varepsilon < \|x^{m_k} - x^{n_k}\| \leq \sum_{s = n_k}^{m_k - 1} \lambda_s \|g^s\| \leq C \sum_{s = n_k}^{m_k - 1} \lambda_s,$$

which yields $\sum_{s = n_k}^{m_k - 1} \lambda_s \geq \varepsilon/C$ and, hence,

$$W(x^{m_k}) \leq W(x^{n_k}) - \gamma'\varepsilon/C.$$

Passage to the limit as $k \longrightarrow \infty$ produces

$$\limsup_{k \to \infty} W(x^{m_k}) < \liminf_{k \to \infty} W(x^{n_k}),$$

which proves Condition 3 with $p_k = m_k$. Since Conditions 4 and 5 obviously hold, this completes the proof of the theorem.

In Theorems 1 and 2, we used the global condition that $\{x^s\}$ is bounded, which is, at first glance, rather restrictive and difficult to check. However, the basic algorithmic scheme (3) can easily be modified using some retracts $R : E \longrightarrow \tilde{V}$ that, if V is bounded, return $\{x^s\}$ into a bounded set $\tilde{V}$ such that $\tilde{V} + U \subset V$:

$$x^{s+1} = \tilde{F}(x^s + z^s) = \begin{cases} F(x^s + z^s), & x^s \in \tilde{W} \supset V \\ R(x^s) = y^s \in V & \text{otherwise}, \end{cases} \tag{7}$$

where y^s is an arbitrary element from $\tilde{V}$. For a locally strongly Fejer F, it is easy to show that the process $\{x^s\}$ leaves $\tilde{W}$ only a finite number of times and, hence, $\{x^s\}$ is bounded. Moreover, $\tilde{F}$ is continuous on $\tilde{W}$. The other aspects of the theory of processes (7) are also covered by Theorem 2.

As in the case of Theorem 1, we have the following result.

Corollary 2. Let F_s be a locally strongly Fejer operator at x^s chosen from a finite family $\mathcal{F} = \{P_1, \ldots, P_m\}$ of continuous operators P_i such that $P_i(v) = v$ $(i = 1, 2, \ldots, m)$ for all $v \in V$ and, for any $x \notin V$, there exists P_i that is locally strongly Fejer at x; $z^s \longrightarrow 0$ as $s \longrightarrow \infty$; Φ be a bounded strong attractant of $Z \subset V$ that is upper semicontinuous on some open set $\tilde{V} \supset V$; and the sequence $\{x^s\}$ be generated by

$$x^{s+1} = F_s(x^s + \lambda_s \Phi(x^s)), \tag{8}$$

where x^0 is an arbitrary initial state, $\lambda_s \longrightarrow +0$, and $\Sigma\lambda_s = \infty$. Then, if $\{x^s\}$ is bounded, any of its limit points belongs to Z.

This assertion follows from Corollary 2 and the uniform continuity of the finite family $\mathcal{F}$ of continuous operators P_i, $i = 1, 2, \ldots, m$.

3. APPLICATION: THE GRADIENT PROJECTION METHOD WITH DECOMPOSITION OF THE CONSTRAINT SYSTEM

In practice, the above results can be applied as follows. On the one hand, the wide range of Fejer operators can be used to solve feasibility problems, which guarantee that the limit points of the constructed sequence $\{x^s\}$ belong to the feasible set V. On the other hand, these results can be used to improve a feasible point so as to make it closer to a set Z of distinguished points in V. Examples are points that solve an optimization or other problem on V. In this case, many methods are available for determining, say, relaxation directions in which the current feasible point approaches the solution set. The essence of Theorem 2 is that, under its assumptions, schemes for finding feasible points and algorithms for achieving the distinguished subset can be fairly easily combined.

In what follows, we consider the important special case where the feasible set V can be represented as the intersection of the family of sets V_τ, $\tau \in T$; i.e., $V = \bigcap_{\tau \in T} V_\tau$. For finite $T = \{1, 2, \ldots, N\}$, numerous Fejer-type algorithms were proposed for searching for a feasible point $v \in V$, i.e., for solving the convex feasibility problem (CFP). Many of these algorithms can be described by an operator of the form

$$F(x) = \sum_{i=1}^{N} w_i[(1-\lambda_i)I + \lambda_i \Pi_i(x)], \tag{9}$$

where Π_i is the projector onto the corresponding V_i, I is the identity operator, $\lambda_i \in [0, 2]$ are relaxation parameters, and $(w_i, i = 1, 2, \ldots, N) = w \in \Delta_N$ are weights of the projections. Let Δ_N denote the standard N-dimensional simplex

$$\Delta_N = \left\{ w_i \geq 0, i = 1, 2, \ldots, N; \sum_{i=1}^{N} w_i = 1 \right\}.$$

These operators make it possible to use the structure of various V_i such that the projection operations onto them are much easier than onto V. Since individual projections Π_i can be calculated regardless of each other, we can use parallel computations, etc.

A general convergence theory of Fejer-type processes based on such operators was presented in [6]. The main result of that theory is that the limit points of the sequence $\{x^s\}$ generated by a recurrence of the form $x^{s+1} = F(x^s)$, $s \longrightarrow \infty$, belong to V for various suitably chosen λ_i and w_i. In some special cases, it can be shown that $x^s \longrightarrow \Pi_V(x^0)$ is the projection of the initial point x^0 onto V. No methods are available for finer tuning.

For simplicity, we consider the operator F_s given by (9), where, for every s, only one weighting coefficient w_i is nonzero and all the relaxation coefficients λ_i are equal to unity. Let us show that, if $x^s \notin V_i$ for some $i = i_s$, then $F_s = \Pi_{i_s}$ is locally strongly Fejer at x^s. This assertion can be formulated as the following lemma.

Lemma 1. *Let V be a convex closed bounded set and W be a convex closed superset of W ($V \subset W$). Then, for $x \notin W$, the projector $\Pi_W(x)$ defined as*

$$\Pi_W(x) \in W, \quad \|\Pi_W(x) - x\| = \min_{w \in W} \|w - x\|,$$

is a locally strongly Fejer operator with respect to V.

Proof. We fix $x \notin W$ and choose a sufficiently small neighborhood U of zero such that $(x + 2U) \cap W = \emptyset$ and $\|\Pi_W(z)\| \le 2\|\Pi_W(x)\| = C/2$ for $z \in x + 2U$. Then, for $z \in x + U$ and arbitrary $v \in V$,

$$\frac{\|\Pi_W(z) - v\|^2}{\|z - v\|^2} \le \frac{\|\Pi_W(z) - v\|^2}{\|z - \Pi_W(z)\|^2 + \|\Pi_W(z) - v\|^2}$$

$$\le 1 - \frac{\|z - \Pi_W(z)\|^2}{\|z - \Pi_W(z)\|^2 + 2(\|\Pi_W(z)\|^2 + \|v\|^2)} \le 1 - \frac{\|z - \Pi_W(z)\|^2}{\|z - \Pi_W(z)\|^2 + 2(C^2/4 + R^2)}$$

$$\le 1 - \frac{\min_{z \in x+U} \|z - \Pi_W(z)\|^2}{\max_{z \in x+U} \|z - \Pi_W(z)\|^2 + 2(C^2/4 + R^2)} = 1 - \theta^2 \le \gamma^2 < 1,$$

where $R \ge \|v\|$ for all $v \in V$. Therefore, $\|\Pi_W(z) - v\| \le \gamma\|z - v\|$ for all $z \in x + U$ and $\gamma < 1$, as required.

Corollary 2 immediately implies that the operator F_s constructed by choosing at the point x^s the operator $F_s = \Pi_{i_s}$ with $x^s \notin V_{i_s}$ guarantees the convergence of the simple iteration $x^{s+1} = F_s(x^s)$ as applied to the CFP. Note that the method for choosing V_{i_s} is of no importance. Therefore, in terms of convergence theory, nearly all the row-action methods [10], such as cyclic projection, farthest set projection, intermittent methods, maximum residual, etc., are covered by Corollary 2.

Moreover, Corollary 2 provides an additional advantage due to the attractant $\Phi(x)$. As an example, we consider the convex mathematical programming problem

$$f_* = \min_{x \in V} f(x) = f(x^*), \quad x^* \in Z \subset V, \tag{10}$$

where the convex closed bounded feasible set V is represented as the intersection of a family of convex closed supersets V_i, $i = 1, 2, \ldots, N$: $V = \bigcap_{i=1,2\ldots,N} V_i$. To direct iterative process (8) toward the solution set of optimization problem (10), it is sufficient to use a special disturbance $z^s = -\lambda_s g^s$, where $g^s \in \Phi(x^s) = \partial f(x^s)$ is an arbitrary subgradient of the objective function in problem (10) that is chosen from the subdifferential set $\partial f(x^s)$ and $\lambda_s \longrightarrow +0$.

Since $0 \le f(x) - f_* \le g(x - z)$ for $g \in \partial f(x)$ and $x \in V \cap Z$, the mapping $\partial f(\cdot)$ is an upper semicontinuous bounded attractant for Z.

Corollary 2 implies the convergence of various alternating gradient projections onto the decomposition elements of V:

$$x^{s+1} = \Pi_s(x^s - \lambda_s g^s), \quad s = 0, 1, \ldots,$$

where Π_s is the projector onto a set V_{i_s} such that $x^s \notin V_{i_s}$), $g^s \in \partial f(x^s)$, f is a finite convex function, $\lambda_s \longrightarrow +0$, and $\Sigma_{s=0}^{\infty} \lambda_s = \infty$.

Using Lemma 1, we can show that the operator

$$F(x) = \sum_{i=1}^{N} w_i \Pi_i(x), \tag{11}$$

where Π_i is the projection onto V_i and $w = (w_1, \ldots, w_N) \in \Delta_N$, is a locally strongly Fejer operator.

Lemma 2. *Let the operator F given by* (11) *be such that* $\Sigma_{i: x \notin V_i} w_i \geq \gamma > 0$. *Then*, $F(x)$ *is a locally strongly Fejer operator at the point x.*

Proof. Let $\bar{x} \notin V$ be a fixed point. Define $I(\bar{x}) = \{i : \|\Pi_i(\bar{x}) - \bar{x}\| > 0\}$. By construction, each Π_i ($i \in I(\bar{x})$) is locally strongly Fejer at $\bar{x}$ with some constant $\alpha_i \leq \alpha < 1$. A neighborhood U of zero can be chosen so that, for some $\delta > 0$ and $\varepsilon > 0$ for all $x \in \bar{x} + U$, we have

$$\|\Pi_i(x) - x\| \geq \delta > 0, \quad i \in I(\bar{x}), \quad \text{and} \quad \|\Pi_i(x) - x\| \leq \varepsilon, \quad i \notin I(\bar{x}),$$

where $\varepsilon < \gamma(1 - \alpha)\delta/2$.

Let $v \in V$ and $x \in \bar{x} + U$. Then,

$$\|F(x) - v\| = \left\|\sum_{i=1}^{N} w_i(\Pi_i(x) - v)\right\| \leq \left\|\sum_{i \in I(\bar{x})} w_i(\Pi_i(x) - v)\right\| + \left\|\sum_{i \notin I(\bar{x})} w_i(\Pi_i(x) - v)\right\|$$

$$\leq \alpha\gamma\|x - v\| + \left\|\sum_{i \notin I(\bar{x})} w_i(x - v)\right\| \leq \alpha\gamma\|x - v\| + (1 - \gamma)\|x - v\|$$

$$\leq (\alpha\gamma + 1 - \gamma)\|x - v\| = \bar{\alpha}\|x - v\|,$$

where $\bar{\alpha} = 1 - \gamma(1 - \alpha) < 1$. That was to be proved.

Due to this result, for problem (10), we can substantiate the use of a parallel version of the gradient projection decomposition method with a Fejer operator of the form (11)

$$x^{s+1} = \sum_{i=1}^{N} w_i^s \Pi_i(x^s - \lambda_s g^s),$$

where the projections can be performed simultaneously. The conditions imposed on the weights w_i^s in Lemma 2 are satisfied, for example, when all of them are uniformly bounded away from zero: $w_i^s \geq \varepsilon > 0$. Accordingly, for step multipliers λ_s, it is sufficient that $\lambda_s \longrightarrow +0$, $\Sigma_{s=0}^{\infty} \lambda_s = \infty$, which are traditional conditions for nonsmooth gradient schemes, although they lead to rather slow convergence.

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