Automation and Remote Control, Vol. 65, No. 2, 2004, pp. 311–318. Translated from Avtomatika i Telemekhanika, No. 2, 2004, pp. 170–178.

Portfolio Replication: Its Forward-Dual Decomposition

A. S. Velichko and E. A. Nurminskii

Institute of Automation and Control Processes, Far-Eastern Branch, Russian Academy of Sciences, Vladivostok, Russia

Received June 28, 2003

Abstract—Replication of a portfolio of market assets under a conditional mean loss criterion is studied. This problem with a risk constraint as the conditional mean loss is studied as a structural extremal problem with binding variables and two groups of constraints. For a large number of assets and continual planning horizons, special methods based on the forward-dual decomposition algorithms are fruitful. Results of numerical experiments are given.

1. INTRODUCTION

Operations in financial (exchange) markets consist in investing free monetary resources for obtaining income from purchase and sale of market assets. These assets are characterized by different yield indexes defined by a long-range trend in the behavior of market prices and different degrees of volatility. Variability and unpredictability of future prices create for portfolio investors risks, which must be controlled. The traditional measures of risk are variance of the portfolio [1] and quantile criterion *VaR* (*Value-at-Risk*). At present, conditional mean loss *CVaR* (*Conditional Value-at-Risk*) criteria are widely used. Risk control is used not only in financial models, in particular, the quantile criterion in investment problems [2], as well as in two-stage stochastic programming [3] and aerospace applications [4].

In investment problems, the main concept is the portfolio, which is described by a vector $x \in X \subseteq \mathbb{R}^n$, whose ith component defines the number of units of asset i, $i = 1, \dots, n$, in the portfolio. The stochastic component of the model is included in the vector y of market prices of assets. This component is a random variable of the set $Y \subseteq \mathbb{R}^n$ on which a probabilistic measure $\mathcal{P}$ is defined. Economic losses are described by a function $f(x, y)$ defining the loss for a given portfolio x and observed price quotation $y \in Y$.

Let the loss distribution function be $\Psi(x,\xi) \triangleq \mathcal{P}\{y | f(x,y) \leqslant \xi\}$ for a given loss function $f(x, y)$ and probabilistic measure $\mathcal{P}$. By definition, the quantile loss function is

$$VaR_\alpha(x) \triangleq \min\{\xi \mid \Psi(x,\xi) \geqslant \alpha\},$$

where $\alpha \in (0, 1)$ is a given probability level.

While the value of *VaR* for a given probability α and a chosen portfolio x is defined to be the minimal loss ξ under which the loss probability is not greater than ξ but greater than α, the value of *CVaR* is defined to be the conditional mean loss greater than *VaR*:

$$CVaR_\alpha(x) \triangleq \mathsf{M}\{f(x,y) \mid f(x,y) > VaR_\alpha(x)\}.$$

Let us state the assertions that play a pivotal role in investigations based on the use of *CVaR*. Their proofs are given in [5].

0005-1179/04/6502-0311 © 2004 MAIK “Nauka/Interperiodica”

Theorem 1. *For $\xi \in \mathbb{R}$, let a scalar function be defined by the expression*

$$F_\alpha(x, \xi) \triangleq \xi + \frac{1}{(1-\alpha)} \mathsf{M}[f(x, y) - \xi]^+ \tag{1}$$

and let $[t]^+ \triangleq \max\{0, t\}$. Then $CVaR_\alpha(x) = \min\limits_{\xi} F_\alpha(x, \xi)$.

A wide class of financial problems is represented by convex loss functions $f(x, y)$ of x. An important property of the *CVaR* criterion is the convexity of the function $CVaR_\alpha(x)$ in x and convexity of the function $F_\alpha(x, \xi)$ (1) in $(x, \xi) \in X \times \mathbb{R}$ for the convex loss functions $f(x, y)$ in x. In this case, the minimization of the conditional mean loss $CVaR_\alpha(x)$ for x is equivalent to the minimization of the function $F_\alpha(x, \xi)$ on $(x, \xi) \in X \times \mathbb{R}$.

The following theorem is important in practical problems with a constraint on the admissible risk level in the conditional mean loss *CVaR* [5].

Theorem 2. *Let $g(x)$ be a scalar function defined for $x \in X \subseteq \mathbb{R}^n$ and let $F_\alpha(x, \xi)$ be a scalar function* (1) *for $\xi \in \mathbb{R}$. Then, for given probability level $\alpha \in (0, 1)$ and admissible loss level $\omega \in \mathbb{R}$, the problem*

$$\min_{x \in X} g(x) \quad \textit{if} \quad CVaR_\alpha(x) \leqslant \omega$$

is equivalent to the problem

$$\min_{x \in X, \xi \in \mathbb{R}} g(x) \quad \textit{if} \quad F_\alpha(x, \xi) \leqslant \omega.$$

The properties of the *VaR* and *CVaR* criteria and their application in financial problems are described in [5, 6].

2. PORTFOLIO REPLICATION

The problem of portfolio replication [5, 7]) is encountered when an investor desires to fix *ex ante* the portfolio structure in some planning horizon so that the portfolio cost at the final instant, if future prices of market assets are known, is equal to a given cost. The investor minimizes his loss. By loss, we mean the measure of difference between the market price of his portfolio and the cost of a reference portfolio consisting only of standard assets. An additional constraint may restrict the maximal loss to be less than a predefined measure. In this paper, this measure is defined by the conditional mean loss corresponding to the *CVaR* criterion.

Let the dynamics of price I_t of the standard asset be known. The investor can buy S_j assets. Their price dynamics is defined by a set of time series p_{tj}, $t = 1, \ldots, T, j = 1, \ldots, n$. Let ν be the planned cost of the portfolio at the final instant T. Then $\theta = \nu / I_T$ is the number of units of the standard asset in the reference portfolio consisting only of this asset of price ν at the instant T. The parameters θI_t describe the cost dynamics of such a portfolio at intermediate instants $t = 1, \ldots, T$. Let x_j be the number of units of assets S_j in the investor's portfolio in the period $[1, T]$. Then $\sum\limits_{j=1}^{n} p_{tj} x_j$ is the cost of the portfolio at the instant t. The loss at the instant t is defined by

$$f_t(x, p) = \left(\theta I_t - \sum_{j=1}^{n} p_{tj} x_j \right) \Big/ \theta I_t,$$

which is the relative difference between the cost of the portfolio of market assets at the instant t and the cost of a portfolio consisting only of standard assets at instant t. This problem describes the desire of the investor to learn in the interval $[1, T]$ how to use the optimal portfolio of a given cost ν with minimal loss in the future ν.

The vectors $p_1, p_2, \ldots, p_T$ are the antecedent costs, which can be regarded as a sample from some general population. Using this information, if there is no hypothesis on the nature of random processes, we can construct only an empirical distribution function and write (1) approximately as

$$F_\alpha(x,\xi) = \xi + \frac{1}{(1-\alpha)}\widetilde{\mathsf{M}}[f(x,p_t)-\xi]^+,$$

where $\widetilde{\mathsf{M}}$ is the expectation for the empirical distribution function.

For stationary and statistically independent cost vectors, the empirical distribution function is concentrated on atoms $\{p_1, p_2, \ldots, p_T\}$ with identical weights $1/T$. Moreover,

$$\widetilde{\mathsf{M}}[f(x,p_t)-\xi]^+ = \frac{1}{T}\sum_{t=1}^{T}[f(x,p_t)-\xi]^+.$$

Therefore,

$$F_\alpha(x,\xi) = \xi + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T}\left[\left[\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right)\Big/\theta I_t\right] - \xi\right]^+.$$

The problem of the investor is to choose a portfolio of assets x_j such that

(1) the mean absolute loss $g(x) = (1/T)\sum\limits_{t=1}^{T}|f_t(x,p)|$ is minimal,

(2) the portfolio cost at the final instant is ν, and

(3) the conditional mean loss in the sense of the *CVaR* criterion for a given probability level α is not greater than a given ω.

According to Theorem 2, the condition $CVaR_\alpha(x) \leqslant \omega$ in the problem of minimization of the function $g(x)$ can be replaced by the constraint $F_\alpha(x,\xi) \leqslant \omega$ in the problem of minimization of $g(x)$ for the variables x and ξ.

Thus, the problem of the investor is to minimize the functional

$$g(x) = (1/T)\sum_{t=1}^{T}\left|\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right)\Big/\theta I_t\right| \tag{2}$$

for x and ξ under the constraints

$$\xi + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T}\left[\left[\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right)\Big/\theta I_t\right] - \xi\right]^+ \leqslant \omega, \tag{3}$$

$$\sum_{j=1}^{n} p_{Tj}x_j = \nu,\ x_j \geqslant 0. \tag{4}$$

In what follows, we take $\beta = 1/[(1-\alpha)T]$. Introducing the variables

$$\eta_t = \left|\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right)\Big/\theta I_t\right| \geqslant 0,$$

$$s_t = \left[\left[\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right)\Big/\theta I_t\right] - \xi\right]^+ \geqslant 0,$$

we reduce problem (2)–(4) to the linear programming problem

$$\min_{x_j,\xi,\eta_t,s_t} (1/T)\sum_{t=1}^{T}\eta_t, \tag{5}$$

$$\eta_t \geqslant \left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right) \Big/ \theta I_t \geqslant -\eta_t, \tag{6}$$

$$\left[\left(\theta I_t - \sum_{j=1}^{n} p_{tj}x_j\right) \Big/ \theta I_t\right] - \xi - s_t \leqslant 0, \tag{7}$$

$$\xi + \beta\sum_{t=1}^{T} s_t \leqslant \omega, \sum_{j=1}^{n} p_{Tj}x_j = \nu \tag{8}$$

under the condition that the variables x_j, η_t, and s_t are nonnegative.

Problem (5)–(8) contains $(3T+2)$ constraints and $(2T+n+1)$ variables. The problem of the investor in a developed financial market is to invest free financial resources in a large number of assets, whose cost may vary every minute. Since the dimension of the linear programming problem is large, we must apply methods that do not require a large memory and time for solution.

3. DECOMPOSITION OF THE PORTFOLIO REPLICATION PROBLEM

Decomposition schemes reduce problem (5)–(8) to subproblems of lesser dimension. These subproblems are modified and intermediate results are exchanged between them. Such schemes have been developed since the beginning of the sixties of the last century [8]. Parallel computation techniques have paved the way for applying decomposition methods from a new standpoint.

The forward-dual cutting algorithm of [9] effectively decomposes a two-block linear programming problem with a relatively small number of connecting variables. In [10], this approach is modified, widening the class of linear programming problems solvable by this algorithm. The forward-dual cutting algorithm can be regarded as the joint application of the Dantzig–Wolf algorithm to forward and dual problems, in each of which the optimized functional is approximated by linear truncations obtained in the course of the operation of the algorithm. The forward and dual problems are modified with regard for the new truncations and intermediate solutions are exchanged between them for improving the solution of the initial problem. This algorithm admits parallel solution of forward and dual problems.

The base structured optimization problem is defined by the two-block problem

$$\min_{z_A,z_B,x} \quad c_A z_A + c_B z_B, \tag{9}$$

$$A_A z_A + B_A x \leqslant d_A, \tag{10}$$

$$A_B z_B + B_B x \leqslant d_B, \tag{11}$$

$$z_A \geqslant 0, \quad z_B \geqslant 0, \quad x \geqslant 0. \tag{12}$$

For a fixed x, this problem decomposes into two independent blocks, which are used by the forward-dual cutting algorithm. The variables x are called the connecting variables since they are contained in all constraints of the problem, whereas z_A and z_B are referred to as internal variables.

Let us show that problem (5)–(8) can be expressed in the form (9)–(12). Since the variable ξ is arbitrary in sign, let us express it in equivalent form as $\xi = \xi_1 - \xi_2 \cdot \xi_1 \cdot \xi_2 \geqslant 0$. The variables η_t form a block of variables z_A and the variables ξ_1, ξ_2, s_t form a block of variables z_B of length $T+2$.

Let P denote a $T \times n$ matrix consisting of elements $p_{tj}/(\theta I_t)$, where e is a unit vector and E is a unit matrix.

Thus, the two-block representation of problem (5)–(8) is reduced to the form (9)–(12)

$$\min\{(\underbrace{1/T, \dots, 1/T}_{T})\eta_t + (\underbrace{0, \dots, 0}_{T+2})(\xi_1, \xi_2, s_t')'\}, \tag{13}$$

$$\begin{pmatrix} -E \\ -E \end{pmatrix} \eta_t + \begin{pmatrix} -P \\ P \end{pmatrix} x \leqslant \begin{pmatrix} -e \\ e \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} e & -e & E \\ 1 & -1 & \beta \dots \beta \\ 0 & 0 & 0 \dots 0 \end{pmatrix} (\xi_1, \xi_2, s_t')' + \begin{pmatrix} & P & \\ 0 & \dots & 0 \\ p_{T1} & \dots & p_{Tn} \end{pmatrix} x \leqslant \begin{pmatrix} e \\ \omega \\ \nu \end{pmatrix}, \tag{15}$$

$$\eta_t, \xi_1, \xi_2, s_t, x \geqslant 0. \tag{16}$$

In the general two-block problem (9)–(12), functions are defined by

$$f_A(x) \triangleq \min_{\substack{z_A \\ A_A z_A \leqslant d_A - B_A x \\ z_A \geqslant 0}} c_A z_A, \quad f_B(x) \triangleq \min_{\substack{z_B \\ A_A z_B \leqslant d_B - B_B x \\ z_B \geqslant 0}} c_B z_B. \tag{17}$$

Then (9)–(12) can be expressed in equivalent form as

$$\min_x \{f_A(x) + f_B(x)\}. \tag{18}$$

The functions $f_A(x))$ and $f_B(x)$ are convex and piecewise-linear. Defining the functions $h_A(p)$ and $h_B(p)$ by the conjugates of $f_A(x)$ and $f_B(x)$, respectively, through the formulas

$$h_A(p) \triangleq f_A^*(x) = \max_x \{px - f_A(x)\},$$
$$h_B(p) \triangleq f_B^*(x) = \max_x \{px - f_B(x)\},$$

we can express (18) in equivalent form by conjugate functions

$$\min_p \{h_A(-p) + h_B(p)\}. \tag{19}$$

Such an equivalence between problems (18) and (19) is helpful in organizing the exchange of coordinating information between the forward and dual linear programming problems in the forward-dual decomposition approach applied to solve problems (9)–(12).

This method consists in replacing the function $f_B(x)$ in (18) and the function $h_A(p)$ in (19) by their outer piecewise-linear approximations obtained in the course of solving subproblems and exchange of forward-dual information. Solving the approximate variants of problems (18) and (19), we obtain the values of the functions $f_B(x)$ and $h_A(p)$ and their subgradients. Thus we obtain new linear truncations to be added to the forward and dual problems, which refine the piecewise-linear approximations of $f_B(x)$ and $h_A(p)$. Computational aspects of the forward-dual cutting algorithm are described in [9, 10].

4. NUMERICAL EXPERIMENTS

In numerical experiments, we used the data on the prices of 10 assets (AA, GE, JNJ, MSFT, AXP, GM, JPM, PG, BA, HD) from 30 assets in the Dow Jones Industrial (DJI) index for the

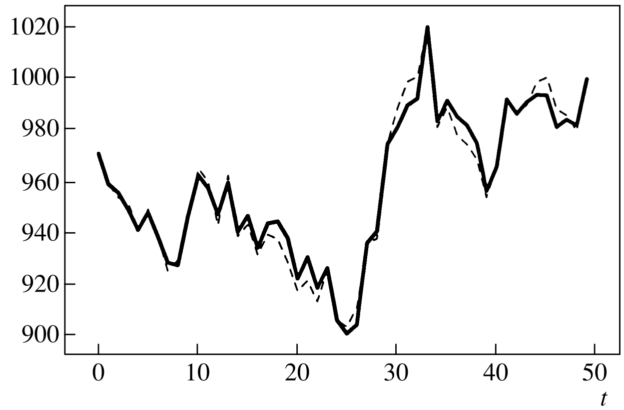

Fig. 1. Dynamics of the cost of the optimal portfolio of the investor x^* for test 7 (dotted line) and reference portfolio (solid line).

period from February 3 to April 14, 2003 (61 quotations for each asset), which were taken from the site at http://finance.yahoo.com. The closing prices were taken to be the market prices of assets in the portfolio and the DJI index for the above period was taken to be the standard asset.

In all experiments, the portfolio terminal cost ν was taken equal to 1000 and a 0.9 probability level α was used in determining $CVaR$. The portfolio replication problem was solved by a modified forward-dual truncation algorithm [10] that is applicable to the structured problem (13)–(16) and by the simplex method that is applicable to the problem (13)–(16) without partitioning into blocks.

For numerical computations, we used a two-processor Pentium III, 800 MHz computer. First we used fifty time periods and the first fifty quotations of the ten 10 assets and standard asset, beginning from March 3, 2003, and admissible loss level ω equal to 0.8. The optimal vector x^* of assets in the portfolio was equal to (1.11, 5.15, 0, 4.58, 4.82, 0.28, 0, 2.49, 2.79, 8.03). The constraints $CVaR_\alpha(x) \leqslant \omega$ on the admissible level of $CVaR$ in this case was inactive since $CVaR_{0.9}(x^*) = 0.009925$. Figure 1 shows the dynamics of the portfolio cost x^*, i.e., $\sum_{j=1}^{10} p_{tj}x_j^*$ for $t = 1, \ldots, 50$ (dotted line), and θI_t defining the dynamics of the reference portfolio (solid line).

Seven more experiments were conducted to measure T for $\omega = 0.8$. The main characteristics of experiments are listed in table, where m denotes the number of constraints in problem (13)–(16), $Na + Nb$ is the total number of internal variables in the blocks z_A and z_B, $k^{\max}$ is the number of forward and dual problems solved by the forward-dual truncation algorithm, MADO is the time of operation of the forward-dual truncation algorithm in seconds, CM is the time of operation of the

Statistics of test examples

Test	T	m	$Na + Nb$	ω	$CVaR_{0.9}(x^*)$	$k^{\max}$	MADO	CM	MADO/CM
1	17	53	36	0.8	0.004616	2	4.55	4.33	1.05
2	21	65	44	0.8	0.003645	1	7.78	10.23	0.76
3	25	77	52	0.8	0.005778	2	16.00	18.95	0.84
4	30	92	62	0.8	0.003240	1	25.93	41.78	0.62
5	35	107	72	0.8	0.006348	2	44.19	62.05	0.71
6	40	122	82	0.8	0.003890	2	76.88	106.36	0.72
7	50	152	102	0.8	0.009925	2	153.17	245.23	0.62
8	60	182	122	0.8	0.007874	2	279.17	470.24	0.59

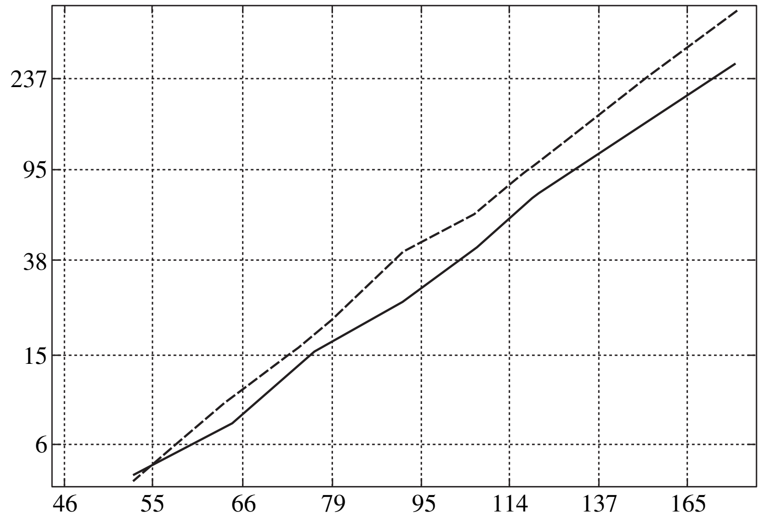

Fig. 2. Time of solution of test example, depending on the number (m) of constraints, by the forward-dual truncation algorithm (solid line) and the simplex method (dotted line). Both axes show the logarithmic scale.

simplex method in seconds in a problem without partitioning into blocks, and MADO/CM is the ratio of times of operation of forward-dual algorithm and simplex method.

With the growth of the planning horizon T defining the dimension of the problem, the gain of the decomposition method increased compared to the solution the problem without any information on the special structure of constraints. Absolute time of machine operation is not of much interest. What matters is the relative gain of the forward-dual algorithm compared to the simplex method without any information on the special structure of constraints. If the optimization packets MINOS and CPLEX are used, the absolute operation time may differ from the tabulated values, but we believe that the relative gain of the forward-dual algorithm will remain the same.

Figure 2 shows time of solution of problem (13)–(16) by the forward-dual truncation algorithm (solid line) and simplex method (dotted line) versus number m of constraints in tests 1-8 characterizing the dimension of the problem. The logarithmic scale is shown in both axes. Since the estimates are linear, the computational complexity of the forward-dual truncation method may be believed to be polynomial. The degree of the polynomial estimate in our tests was equal to 3.39, but the actual complexity of the simplex method applied in a problem without blocks in tests was proportional to $m^{3.77}$. For an m of order 1000, computations are 14 fold faster.

5. CONCLUSIONS

Portfolio replication is studied with the *CVaR* criterion. This problem is reduced to a linear programming problem, in which the mean absolute value of the relative difference between the costs of market asset portfolio and reference portfolio in time is minimized. This problem is solved by the forward-dual decomposition method [9, 10]. Numerical experiments demonstrate that computation cost for the forward-dual truncation method is reduced compared to the standard simplex method.

REFERENCES

1. Markowitz, H.M., Portfolio Selection, *J. Finances*, 1952, no. 7(1), pp. 77–91.
2. Kibzun, A.I. and Kuznetsov, E.A., Optimal Portfolio Control, *Avtom. Telemekh.*, 2001, no. 9, pp. 101–113.
3. Kibzun, A.I. and Nikulin, I.V., A Discrete Approximation for a Linear Two-Stage Stochastic Programming Problem under Quantile Criterion, *Avtom. Telemekh.*, 2001, no. 8, pp. 127–137.

4. Kan, Yu.S., Optimization of Control by the Quantile Criterion, *Avtom. Telemekh.*, 2001, no. 5, pp. 77–88.

5. Rockafellar, R.T., and Uryasev, S., Conditional Value-at-Risk for General Loss Distributions, *J. Banking Finance*, 2002, no. 26, pp. 1443–1471.

6. Rockafellar, R.T. and Uryasev, S., Optimization of Conditional Value-at-Risk, *J. Risk*, 2000, no. 2, pp. 21–41.

7. Dembo, R.S. and Rosen, D., The Practice of Portfolio Replication: A Practical Overview of Forward and Inverse Problem, *Ann. Oper. Res.*, 1999, no. 85, pp. 267–284.

8. Lasdon, L.S., *Optimization Theory for Large Systems*, London: Macmillan, 1969. Translated under the title *Optimizatsiya bol'shikh sistem*, Moscow: Nauka, 1975.

9. Nurminskii, E.A., *Chislennye metody vypukloi optimizatsii* (Numerical Methods of Convex Optimization), Moscow: Nauka, 1991.

10. Velichko, A.S. and Nurminskii, E.A., Decomposition Method of Finite Elements by the Theory of Structured Optimization Problems, *Issledono v Rossii*, 2002, no. 113, pp. 1237–1256.

This paper was recommended for publication by A.I. Kibzun, a member of the Editorial Board