

# Extended Conjugate Space Algorithms for Non-Smooth Optimization

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# Outline

- The high-speed epi-projection optimization solver (EPOS);
- Implementable EPOS;
- Computational results and issues.

# The basic idea of ECSEP Algorithms

The basic idea:

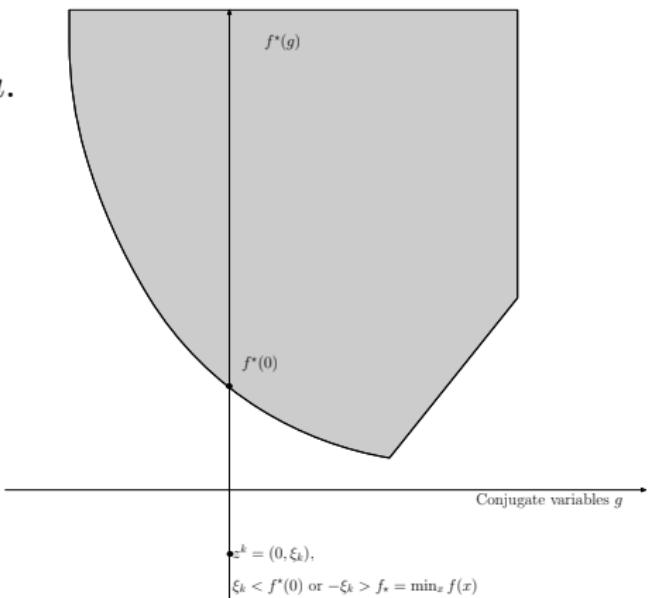
$$\min_x f(x) = -f^*(0) = -\inf_{(0,\mu) \in \text{epi } f^*} \mu.$$

where  $f^*(g) = \sup_x \{xg - f(x)\}$ .

From optimality condition

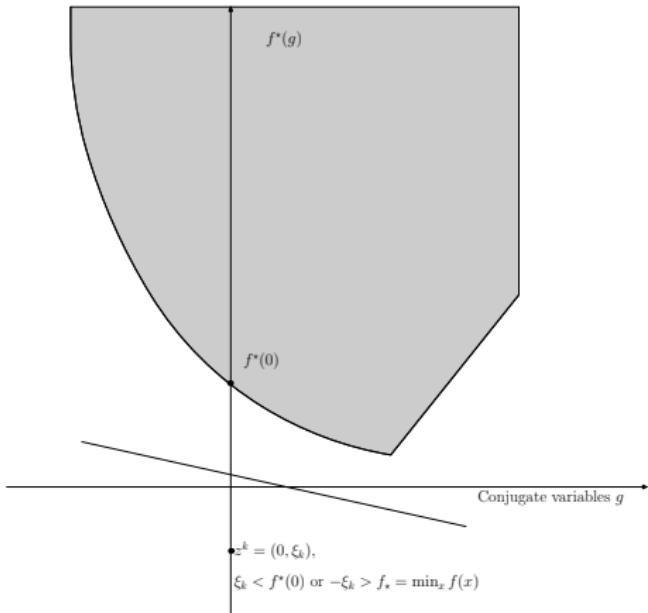
$$0 \in \partial f(x^*), x^* \in \partial f^*(0),$$

$\bar{x}^* = (x^*, -1)$  is a support vector  
to  $\text{epi } f^*$  at the point  $(0, f^*(0)) = (0, -f(x^*))$ .



Can be traced to *A class of convex programming methods*, Dec 1986, USSR Comp Math and Math Phys 26(4):122-128 and *Separating plane algorithms for convex optimization*, Math Programming, 76 (1997), 375–391

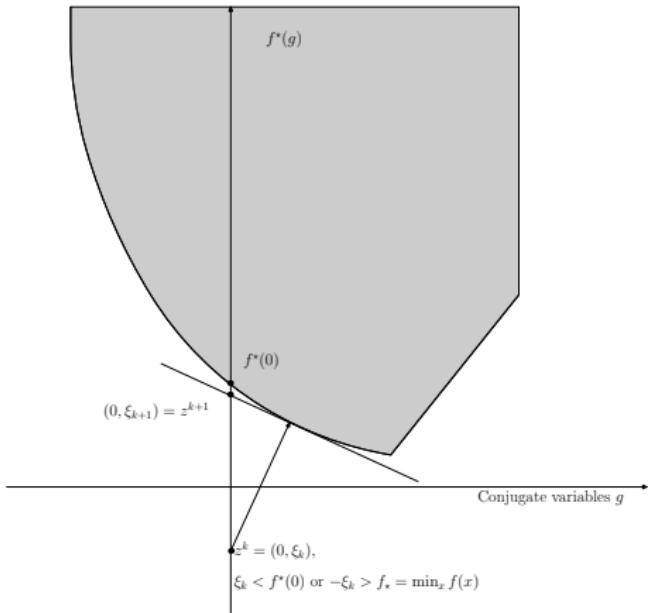
# Separating planes idea



## Separating planes

- Great flexibility;
- Allows to use external approximations of epi  $f^*$ , f.i. gradient-free algorithms, inexact oracles, etc;

# Supporting planes algorithm



## Supporting planes

- Tight outer approximation of  $\text{epi } f^*$ ;
- The auxiliary projection problem well-behaves  $L = 1$ ;
- Fast convergence;

# Epi-Projection Optimization Solver

Guaranteed computational efficiency:

- If  $f(x)$  is just convex the convergence is superlinear:

$$f_{k+1} - f_* \leq \lambda_k(f_k - f_*), \quad \lambda_k \rightarrow 0 \text{ when } k \rightarrow \infty$$

- If  $f(x)$  is sup-quadratic the convergence is quadratic:

$$f_{k+1} - f_* \leq \lambda(f_k - f_*)^2, \quad \text{when } k \rightarrow \infty$$

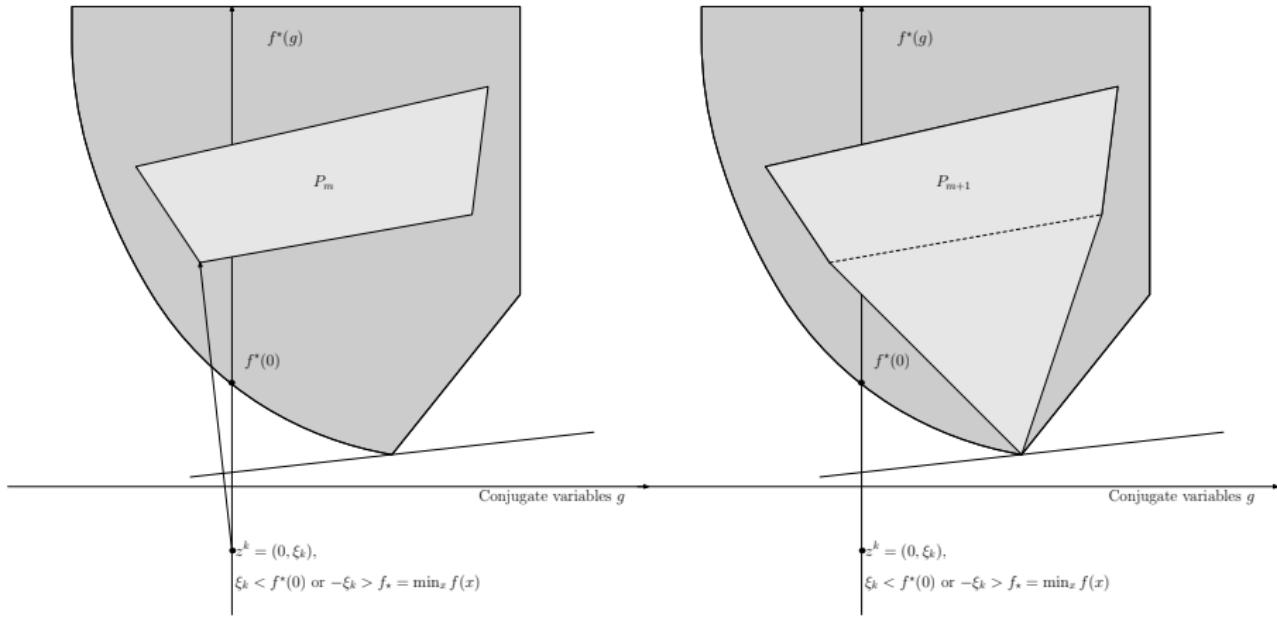
when  $\lambda < f_0 - f_*$  which guarantees convergence.

- If  $f(x)$  has sharp minimum then convergence is finite.

In all cases convergence is global, it does not depend on initial point.

# Implementable version

Based on approximation of epi  $f^*$  with with  
 $P_m = \text{co}\{z^1, z^1, \dots, z^m\}, m = 1, 2, \dots$



# Practicalities

The specialized algorithm PTP for projection on the polytop  $P_m$ , based on *Nurminski, E.A. Convergence of the Suitable Affine Subspace Method for Finding the Least Distance to a Simplex. Computational Mathematics and Mathematical Physics, 45(11), 1915–1922 (2005)*

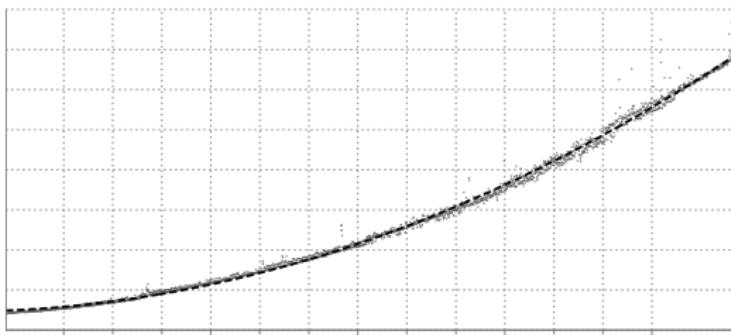
One can be downloaded from DOI: 10.13140/RG.2.2.21281.86882 on ResearchGate. Both PYTHON and OCTAVE versions are available.

**Interesting enough:**

$$\begin{array}{lll} \min \|x\|^2 & \Rightarrow & \min \|x - a\|^2 \\ Ax \leq b & & x \in K_{A,b} \end{array} \Rightarrow \min \|x\|^2 \quad x \in P_{A,b}$$

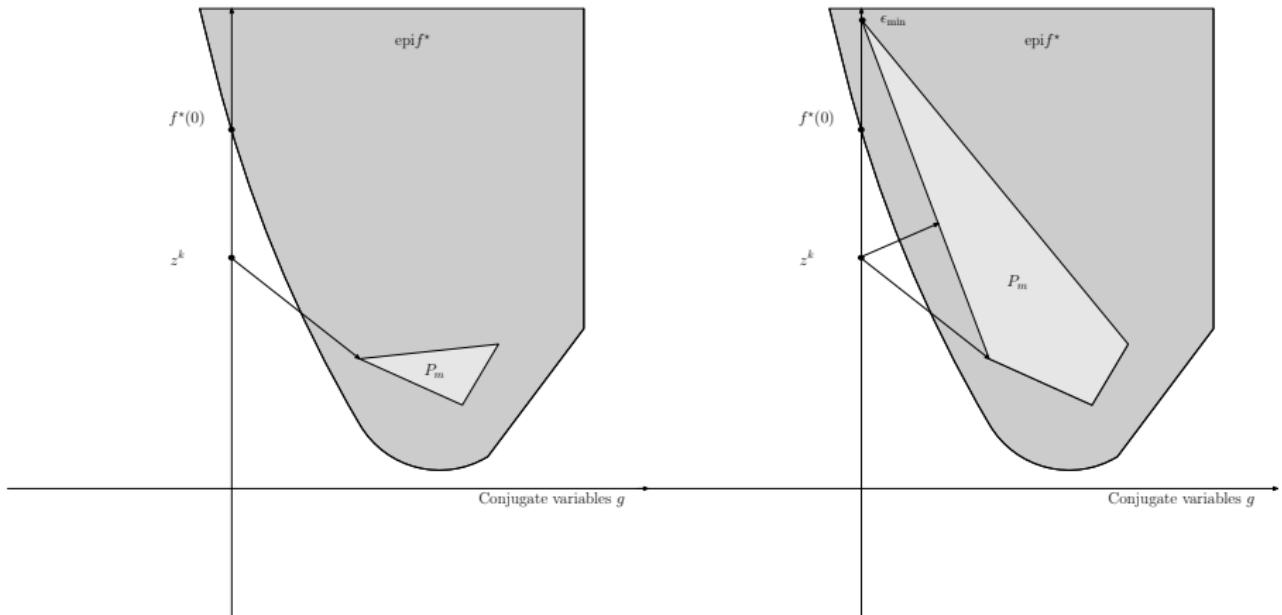
of practically the same data-size.

# PTP iterations complexity



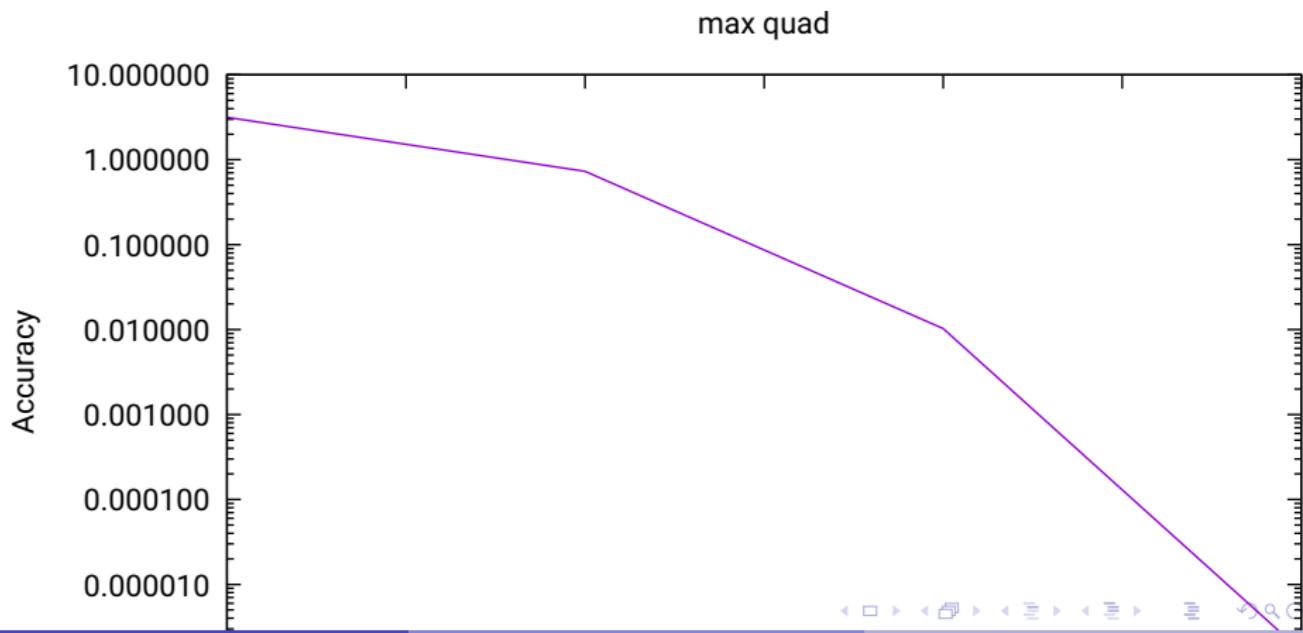
PTP run-time dependence on the base size, fitted with the quadratic approximation  $1.833 \cdot 10^{-8}x^2 + 5.764 \cdot 10^{-6}x + 0.0097$

# Projection problem setup — hard case

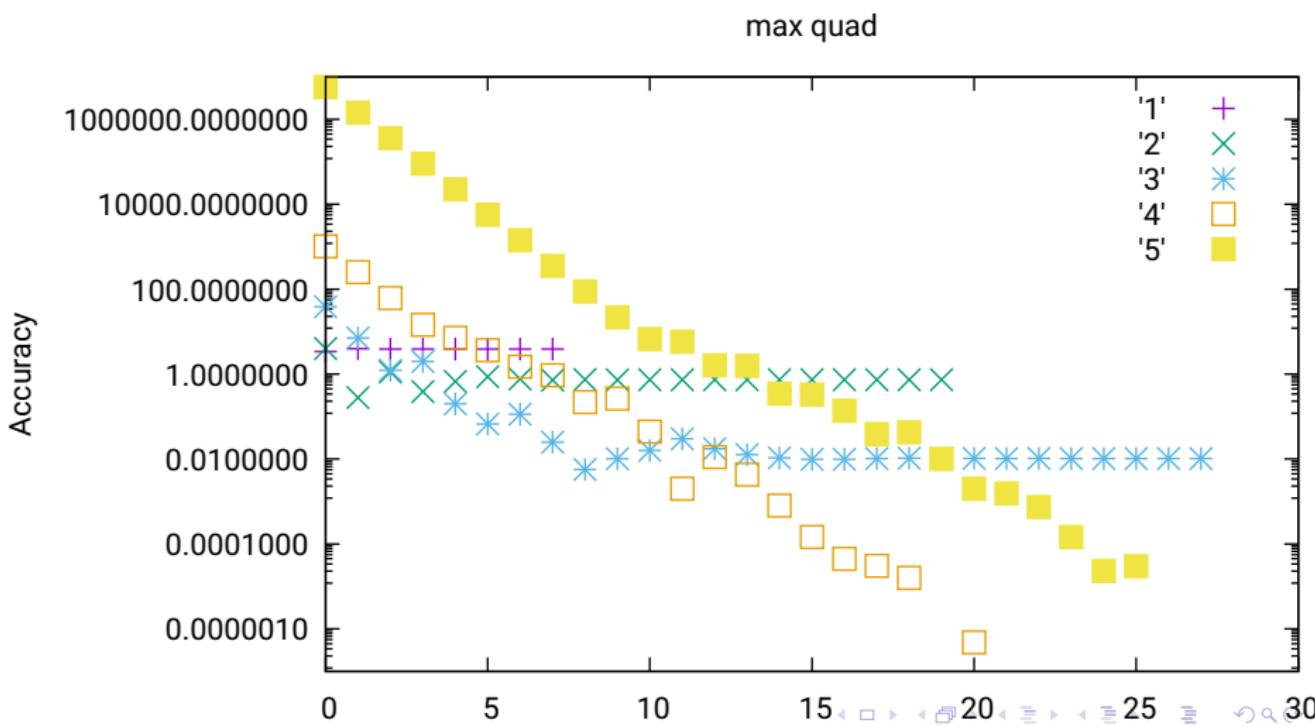


# Piece-wise quadratic test

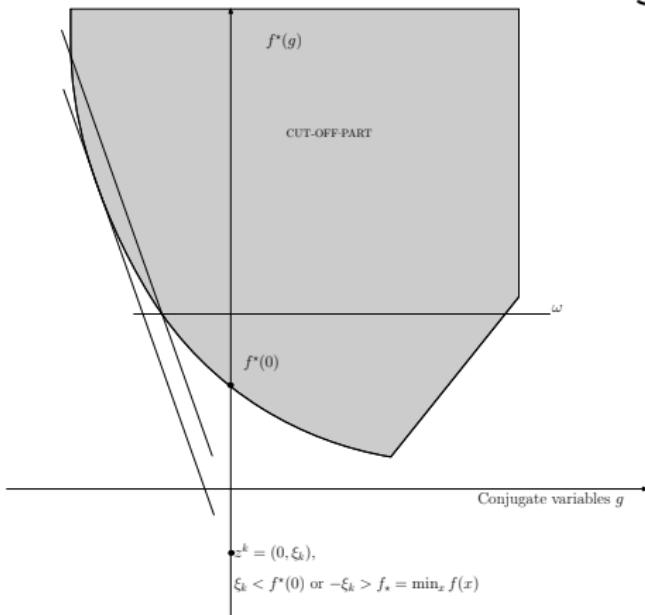
$$f(x) = \max\{q_1(x), q_2(x)\}$$



# Restart Problems



# Down-cut in the Epi-Projection Optimization Solver



Supporting with down-cuts planes

$$\begin{aligned} \min \quad & \{xg - \mu\} = f_\omega(x) \\ \mu \geq f^*(g) \\ \mu \leq \omega \end{aligned}$$

where

$$f_\omega(x) = \inf_{\theta \geq 1} \theta(f(x/\theta) + \omega) - \omega$$

# Interval mathematics and nonsmooth optimization

Interval values (vectors, matrices):  $a = [a_{\min}, a_{\max}]$ . The existence of solution for the  $m \times n$  interval systems of linear equations:  $Ax = b$  is equivalent to minimization of a special function

$$\text{Tol}(x) = \min_{1 \leq i \leq m} \left\{ \text{rad}(b_i) - \text{Abs}(\text{mid}(b_i)) - \sum_{j=1}^n a_{ij}x_j \right\}$$

where

$$\text{mid}(a) = 0.5(a_{\min} + a_{\max}), \quad \text{rad}(a) = a_{\max} - a_{\min}, \quad \text{Abs}(a) = \max\{|a_{\min}|, |a_{\max}|\}$$

OCTAVE/MATLAB notation:

```
f = min( 0.5*(bsup - binf) - max(abs(0.5*(binf + bsup) - sum(max(x' .*Ainf, x' .*Asup),2)), abs(0.5*(binf + bsup) - sum(min(x' .*Ainf, x' .*Asup),2))))
```