

Multiplicative boundary control problems for nonlinear reaction-diffusion-convection model

Zh Saritskaia¹ and P Savinov²

¹ Institute of Applied Mathematics FEB RAS, Vladivostok, Russia

² Far Eastern Federal University, Vladivostok, Russia

E-mail: zhsar@icloud.com, savinov_pa@dvfu.ru

Abstract. Global solvability of a boundary value problem for a generalised Boussinesq model is proved in the case, when reaction coefficient depends nonlinearly on concentration of substance. Solvability of control problem is proved, when the role of controls is played by mass exchange coefficients from the boundary conditions of the model.

1. Introduction. Boundary value problem

During a long period the interest for the studying of boundary value and control problems for linear and nonlinear heat-and-mass transfer models hasn't waned (see [1–15]). In addition to the search of efficient mechanisms of control of physical processes in continuous medium, the study of control problems has also other applications. In the framework of the optimisation approach a number of inverse problems are reduced to control ones. In particular, inverse coefficient problems are reduced to the multiplicative control problems (see [15–19] about the correctness of this approach).

In a current paper, while considering boundary value and extremum problems we suppose that a reaction coefficient depends rather arbitrarily on substance's concentration and on spatial variable. Let us note [20, 21], where a generalisation of Oberbeck-Boussinesq approximation is also used for different models. Let us also mention the papers [22, 23], which are dedicated to the study of equally interesting and complicated hydrodynamical models.

In a bounded domain $\Omega \subset \mathbf{R}^3$ with boundary Γ , consisting of the two parts Γ_D and Γ_N , the following boundary value problem is considered:

$$-\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} + \beta\mathbf{G}\varphi, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1)$$

$$-\operatorname{div}(\lambda(\mathbf{x})\nabla\varphi) + \mathbf{u} \cdot \nabla\varphi + k(\varphi, \mathbf{x})\varphi = f \quad \text{in } \Omega, \quad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \quad \varphi = 0 \quad \text{on } \Gamma_D, \quad \lambda(\mathbf{x})(\partial\varphi/\partial n + \alpha(\mathbf{x})\varphi) = \chi \quad \text{on } \Gamma_N. \quad (3)$$

Here \mathbf{u} is a velocity vector, function φ represents the concentration of the pollutant, $p = P/\rho$, where P is pressure, $\rho = \text{const}$ is the fluid density, $\nu = \text{const} > 0$ is the constant kinematic viscosity, $\lambda = \lambda(\mathbf{x}) > 0$ is the diffusion coefficient, β is the coefficient of mass expansion, $\mathbf{G} = -(0, 0, G)$ is the acceleration of gravity, \mathbf{f} and f are volume densities of external forces or external sources of the substance, the function $k = k(\varphi, \mathbf{x})$ is the reaction coefficient, where



$\mathbf{x} \in \Omega$, $\alpha = \alpha(\mathbf{x})$ is a mass-transfer coefficient. Below we will refer to the problem (1)–(3) for given functions $\mathbf{f}, f, \lambda, \beta, k, \alpha$ and χ as Problem 1.

In this paper we prove a global solvability of Problem 1 and local uniqueness of its solution in the case when reaction coefficient depends rather arbitrarily on substance's concentration φ and on spatial variable $\mathbf{x} \in \Omega$. Furthermore, for the considered boundary value problem a control problem is formulated. The role of controls is played by coefficient α . Solvability of the extremum problem is also proved or reaction coefficients of a general form. Here we will also note the papers [24, 25] on studying related models of complex heat exchange.

Below we will use the Sobolev functional spaces $H^s(D)$, $s \in \mathbf{R}$. Here D means either a domain Ω or some subset $Q \subset \Omega$, or a boundary Γ or its part $\Gamma_0 \subset \Gamma$. By $\|\cdot\|_{s,Q}$, $|\cdot|_{s,Q}$ and $(\cdot, \cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$, respectively. The norms and the scalar product in $L^2(Q)$, $L^2(\Omega)$ and $L^2(\Gamma_N)$ will be denoted, correspondingly, by $\|\cdot\|_Q$ and $(\cdot, \cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot, \cdot)_\Omega$, $\|\cdot\|_{\Gamma_N}$ and $(\cdot, \cdot)_{\Gamma_N}$. Let $L_+^p(D) = \{k \in L^p(D) : k \geq 0\}$, $p \geq 3/2$, $H_{\lambda_0}^s(\Omega) = \{\lambda \in H^s(\Omega) : \lambda \geq \lambda_0 > 0\}$, $s > 3/2$, $L_0^2(\Omega) = \{h \in L^2(\Omega) : (h, 1) = 0\}$.

By $V = \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}$, and $\mathcal{T} = \{h \in H^1(\Omega) : h|_{\Gamma_D} = 0\}$ we introduce the main functional spaces for a velocity vector \mathbf{u} and for concentration φ .

Let the following conditions hold:

(i) Ω is a bounded domain in \mathbf{R}^3 with boundary $\Gamma \in C^{0,1}$, which is a union of the closure of two disjoint open sets Γ_D and of Γ_N ($\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$). Further, the surface measure $\operatorname{meas} \Gamma_D$ is positive, and the boundary $\partial\Gamma_D$ of the set Γ_D consists of finitely many closed Lipschitz curves;

(ii) $\lambda \in H_{\lambda_0}^s(\Omega)$, $s > 3/2$, $\mathbf{f} \in L^2(\Omega)^3$, $f \in L^2(\Omega)$, $\mathbf{b} = \beta \mathbf{G} \in L^2(\Omega)^3$, $\chi \in L^2(\Gamma_N)$;

(iii) $\alpha \in L_+^2(\Gamma_N)$;

(iv) for any function $w \in \mathcal{T}$, the embedding $k(w, \cdot) \in L_+^p(\Omega)$ is true for some $p \geq 3/2$, where p does not depend on w ; and on any sphere $B_r = \{w \in \mathcal{T} : \|w\|_{1,\Omega} \leq r\}$ of radius r the following inequality takes place:

$$\|k(w_1, \cdot) - k(w_2, \cdot)\|_{L^p(\Omega)} \leq L \|w_1 - w_2\|_{L^4(\Omega)} \quad \forall w_1, w_2 \in \mathcal{T}.$$

Here L is the constant, which depends on r , but does not depend on $w_1, w_2 \in B_r$.

Let us note that the condition (iv) describes an operator, acting from \mathcal{T} to $L^p(\Omega)$, where $p \geq 3/2$, which gives us an opportunity to take into consideration the dependence of the reaction coefficient on either the component φ of the solution (\mathbf{u}, φ, p) of Problem 1 or on the spatial variable $\mathbf{x} \in \Omega$ (see [13, 15]).

Below we will just write $k(\varphi)$, while emphasizing the nonlinear dependence of reaction coefficient on the concentration.

Let us also remind that, by the Sobolev embedding theorem, the space $H^1(\Omega)$ is embedded into the space $L^s(\Omega)$ continuously at $s \leq 6$ and compactly at $s < 6$ and, with a certain constant C_s , depending on s and Ω , we have the estimate

$$\|\varphi\|_{L^s(\Omega)} \leq C_s \|\varphi\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega). \quad (4)$$

The following technical lemma holds.

Lemma 1.1. *Under the condition (i) $k_0 \in L_+^p(\Omega)$, $p \geq 3/2$, $\mathbf{u} \in V$, $\mathbf{b} \in L^2(\Omega)^3$, $\lambda \in H_{\lambda_0}^s(\Omega)$, $s > 3/2$, $\alpha \in L_+^2(\Gamma_N)$ there exist positive constants $C_0, C_1, \delta_0, \delta_1, \gamma_1, \gamma_1', \gamma_2, \gamma_2', \gamma_3, \gamma_p, \beta_1$, which depend on Ω or depend on Ω and p , and there is a constant β_0 , which depends on $\|\mathbf{b}\|_\Omega$, such that the following relations hold:*

$$|(\mathbf{b}h, \mathbf{w})| \leq \beta_0 \|h\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{w} \in H^1(\Omega)^3, h \in H^1(\Omega), (\nabla \mathbf{v}, \nabla \mathbf{v}) \geq \delta_0 \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3,$$

$$|((\mathbf{w} \cdot \nabla) \mathbf{h}, \mathbf{z})| \leq \gamma_1' \|\mathbf{w}\|_{L^4(\Omega)^3} \|\mathbf{h}\|_{1,\Omega} \|\mathbf{z}\|_{1,\Omega} \leq \gamma_1 \|\mathbf{w}\|_{1,\Omega} \|\mathbf{h}\|_{1,\Omega} \|\mathbf{z}\|_{1,\Omega} \quad \forall \mathbf{w}, \mathbf{h}, \mathbf{z} \in H^1(\Omega)^3, \quad (5)$$

$$((\mathbf{u} \cdot \nabla) \mathbf{w}, \mathbf{v}) = -((\mathbf{u} \cdot \nabla) \mathbf{w}, \mathbf{v}), ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3,$$

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^3, \mathbf{v} \neq \mathbf{0}} -(\operatorname{div} \mathbf{v}, p) / \|\mathbf{v}\|_{1,\Omega} \geq \beta_1 \|p\|_\Omega \quad \forall p \in L_0^2(\Omega),$$

$$|(\mathbf{u} \cdot \nabla h, \eta)| \leq \gamma_2' \|\mathbf{u}\|_{L^4(\Omega)^3} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega} \leq \gamma_2 \|\mathbf{u}\|_{1,\Omega} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega} \quad \forall h, \eta \in H^1(\Omega);$$

$$|(k_0 h, \eta)| \leq \gamma_p \|k_0\|_{L^p(\Omega)} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega}, (\lambda \alpha h, \eta)_{\Gamma_N} \leq \gamma_3 \|\lambda\|_{s,\Omega} \|\alpha\|_{\Gamma_N} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega} \quad \forall \varphi, \eta \in H^1(\Omega),$$

$$(\mathbf{u} \cdot \nabla h, h) = 0, (\lambda \nabla h, \nabla h) \geq \lambda_* \|h\|_{1,\Omega}^2 \quad \forall h \in \mathcal{T}, \lambda_* \equiv \delta_1 \lambda_0.$$

Let us multiply the first equation in (1) by a function $\mathbf{v} \in H_0^1(\Omega)^3$, the first equation in (2) by a function $h \in \mathcal{T}$ and integrate over Ω using Green's formulae. We obtain the weak formulation of Problem 1:

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \quad (6)$$

$$(\lambda \nabla \varphi, \nabla h) + (k(\varphi)\varphi, h) + (\lambda \alpha \varphi, h)_{\Gamma_N} + (\mathbf{u} \cdot \nabla \varphi, h) = (f, h) + (\chi, h)_{\Gamma_N} \quad \forall h \in \mathcal{T}. \quad (7)$$

The triple $(\mathbf{u}, \varphi, p) \in V \times \mathcal{T} \times L_0^2(\Omega)$, which satisfies (6), (7), will be called a weak solution of Problem 1.

We consider the restriction of (6) on the space V :

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi, \mathbf{v}) \quad \forall \mathbf{v} \in V. \quad (8)$$

To prove the existence of a weak solution of Problem 1 it is enough to prove the existence of a solution $(\mathbf{u}, \varphi) \in V \times \mathcal{T}$ of the problem (7), (8). About the restoration of pressure see more in [5, 14]. In its turn, the proof of the solvability of the problem (7), (8) will be constructed with the help of the Schauder fixed-point theorem, like in [14].

The following theorem holds.

Theorem 1.1. *Assume that the assumptions (i)–(iv) hold. Then there exists a weak solution $(\mathbf{u}, \varphi, p) \in V \times \mathcal{T} \times L_0^2(\Omega)$ of Problem 1, and the following estimates hold:*

$$\|\varphi\|_{1,\Omega} \leq M_\varphi \equiv C_*(\|f\|_\Omega + \gamma_3 \|\chi\|_{\Gamma_N}), \quad \|\mathbf{u}\|_{1,\Omega} \leq M_{\mathbf{u}} = \nu_*^{-1}(\|f\|_\Omega + \beta_0 M_\varphi), \quad C_* = \lambda_*^{-1},$$

$$\|p\|_\Omega \leq C_p = \beta_2^{-1}[(\nu + \gamma_1 M_{\mathbf{u}})M_{\mathbf{u}} + \|f\|_\Omega + \beta_0 M_\varphi], \quad \beta_2 = (\beta_1 - \delta) > 0, \quad \delta > 0.$$

If, besides, the following conditions hold:

$$\operatorname{Re} \leq 1/2, \quad (\gamma_p(\delta_0 \nu / \beta_0 \gamma_2) C_4 L + 2) \operatorname{Ra} < 1,$$

where $\operatorname{Re} = (\gamma_1 / \delta_0 \nu) M_{\mathbf{u}}$ and $\operatorname{Ra} = (\gamma_2 / \delta_0 \nu)(\beta_0 / \delta_1 \lambda) M_\varphi$ are dimensionless analogues of Reynolds number and of diffusion Rayleigh number, then a weak solution of Problem 1 is unique. Here the constants $C_4, \gamma_p, \delta_0, \beta_0, \gamma_2, L$ are defined in (4), Lemma 1.1 and condition (iii), respectively.

2. Multiplicative control problem

In this section we will study a multiplicative control problem for the system (1)–(3), in which the role of the control is played by coefficient α . We assume that α can be changed in subset K , which satisfy the following condition:

(j) $K \subset L_+^2(\Gamma_N)$ are a nonempty convex closed sets.

Define functional spaces $X = H_0^1(\Omega)^3 \times \mathcal{T} \times L_0^2(\Omega)$, $Y = H^{-1}(\Omega)^3 \times \mathcal{T}^* \times L_0^2(\Omega)$ and $\mathbf{x} = (\mathbf{u}, \varphi, p) \in X$, introduce an operator $F = (F_1, F_2) : X \times K \rightarrow Y$ by formulae

$$\langle F_1(\mathbf{x}, u), (\mathbf{v}, h) \rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\lambda \nabla \varphi, \nabla h) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (k(\varphi)\varphi, h) + (\mathbf{u} \cdot \nabla \varphi, h) +$$

$$+(\lambda\alpha\varphi, h)_{\Gamma_N} - (\mathbf{f}, \mathbf{v}) - (\mathbf{b}\varphi, \mathbf{v}) - (f, h), \quad \langle F_2(\mathbf{x}, u), r \rangle = -(\operatorname{div} \mathbf{u}, r)$$

and rewrite a weak form (7) of Problem 1 in the form of the operator equation $F(\mathbf{x}, u) = 0$.

Let $I : X \rightarrow \mathbf{R}$ be a weakly lower semicontinuous functional. Consider the following multiplicative control problem:

$$J(\mathbf{x}, \alpha) \equiv (\mu_0/2)I(\mathbf{x}) + (\mu_1/2)\|\alpha\|_{\Gamma_N}^2 \rightarrow \inf, \quad F(\mathbf{x}, \alpha) = 0, \quad (\mathbf{x}, \alpha) \in X \times K. \quad (9)$$

The set of possible pairs for the problem (9) is denoted by $Z_{ad} = \{(\mathbf{x}, \alpha) \in X \times K : F(\mathbf{x}, \alpha) = 0, J(\mathbf{x}, \alpha) < \infty\}$. Let, in addition to (i), the following condition hold:

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$ and K is a bounded set or $\mu_i > 0$, $i = 0, 1$ and a functional I is bounded from below.

We use the following cost functionals:

$$I_1(\varphi) = \|\varphi - \varphi^d\|_Q^2, \quad I_2(\varphi) = \|\varphi - \varphi^d\|_{1,Q}^2, \quad I_3(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}^d\|_Q^2, \quad I_4(p) = \|p - p^d\|_Q^2. \quad (10)$$

Here a function $\varphi^d \in L^2(Q)$ denotes a desired concentration field, which is given in a subdomain $Q \subset \Omega$. Functions \mathbf{u}^d and p^d have a similar sense for either a velocity field or pressure.

Theorem 2.1. *Assume that the assumptions (i), (ii), (iv) and (j), (jj) take place. Let $I : X \rightarrow \mathbf{R}$ be a weakly semicontinuous below functional and let $Z_{ad} \neq \emptyset$. Then there is at least one solution $(\mathbf{x}, \alpha) \in X \times K$ of the control problem (9).*

Proof. Let $(\mathbf{x}_m, \alpha_m) \in Z_{ad}$ be a minimizing sequence for which the following is true:

$$\lim_{m \rightarrow \infty} J(\mathbf{x}_m, \alpha_m) = \inf_{(\mathbf{x}, \alpha) \in Z_{ad}} J(\mathbf{x}, \alpha) \equiv J^*.$$

From the condition (jj) and from Theorem 1.1 it can be deduced that the estimates hold:

$$\|\alpha_m\|_{\Gamma_N} \leq c_1, \quad \|\mathbf{u}_m\|_{1,\Omega} \leq c_2, \quad \|\varphi_m\|_{1,\Omega} \leq c_3, \quad \|p_m\|_{\Omega} \leq c_4, \quad (11)$$

where the constants c_1, c_2, \dots don't depend on m . From the estimate (11) and from the condition (j) it follows that there exist weak limits $\alpha^* \in K$, $\mathbf{u}^* \in V$, $\varphi^* \in \mathcal{T}$, $p^* \in L_0^2(\Omega)$ of some subsequences of sequences $\{\alpha_m\}$, $\{\mathbf{u}_m\}$, $\{\varphi_m\}$, $\{p_m\}$, respectively. With this in mind, it can be considered that, as $m \rightarrow \infty$, we have

$$\begin{aligned} p_m &\rightarrow p^* \text{ weakly in } L^2(\Omega), \quad \mathbf{u}_m \rightarrow \mathbf{u}^* \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^p(\Omega)^3, \quad p < 6, \\ \alpha_m &\rightarrow \alpha^* \text{ weakly in } L^2(\Gamma_N), \quad \varphi_m \rightarrow \varphi^* \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^s(\Omega), \quad s < 6, \\ \varphi_m|_{\Gamma_N} &\rightarrow \varphi^*|_{\Gamma_N} \text{ weakly in } L^4(\Gamma_N) \text{ and strongly in } L^q(\Gamma_N), \quad q < 4. \end{aligned} \quad (12)$$

It is clear that $F_2(\mathbf{x}^*, \alpha) = 0$. Let us show that $F(\mathbf{x}^*, \alpha^*) = 0$, i.e. that

$$\begin{aligned} &\nu(\nabla \mathbf{u}^*, \nabla \mathbf{v}) + (\lambda \nabla \varphi^*, \nabla h) + ((\mathbf{u}^* \cdot \nabla) \mathbf{u}^*, \mathbf{v}) - (p^*, \operatorname{div} \mathbf{v}) + (k(\varphi^*)\varphi^*, h) + \\ &+ (\mathbf{u}^* \cdot \nabla \varphi^*, h) + (\lambda \alpha^* \varphi^*, h)_{\Gamma_N} = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi^*, \mathbf{v}) + (f, h) \quad \forall (\mathbf{v}, h) \in H. \end{aligned} \quad (13)$$

Let us also remind that (\mathbf{x}_m, α_m) satisfies the relation

$$\begin{aligned} &\nu(\nabla \mathbf{u}_m, \nabla \mathbf{v}) + (\lambda \nabla \varphi_m, \nabla h) + ((\mathbf{u}_m \cdot \nabla) \mathbf{u}_m, \mathbf{v}) - (p_m, \operatorname{div} \mathbf{v}) + (k(\varphi_m)\varphi_m, h) + \\ &+ (\mathbf{u}_m \cdot \nabla \varphi_m, h) + (\lambda \alpha_m \varphi_m, h)_{\Gamma_N} = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi_m, \mathbf{v}) + (f, h) \quad \forall (\mathbf{v}, h) \in H_0^1(\Omega)^3 \times \mathcal{T}. \end{aligned} \quad (14)$$

Let us pass to the limit in (14) at $m \rightarrow \infty$. From (12) it follows that all linear terms in (14) turn into corresponding ones in (13). Let us consider the nonlinear terms, starting with $(k(\varphi_m)\varphi_m, h)$.

From the condition (iii) it follows that $k(\varphi_m) \rightarrow k(\varphi^*)$ strongly in $L^{3/2}(\Omega)$ as $m \rightarrow \infty$. It's not hard to show that from (12) a weak convergence $\varphi_m h \rightarrow \varphi^* h$ in $L^3(\Omega)$ for all $h \in \mathcal{T}$ follows. Then $k(\varphi_m)\varphi_m h \rightarrow k(\varphi^*)\varphi^* h$ strongly in $L^1(\Omega)$ as $m \rightarrow \infty$ or

$$|(k(\varphi_m)\varphi_m, h) - (k(\varphi^*)\varphi^*, h)| \rightarrow 0 \text{ as } m \rightarrow \infty \forall h \in \mathcal{T}.$$

It is clear, that $((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{v}) - ((\mathbf{u}^* \cdot \nabla)\mathbf{u}^*, \mathbf{v}) = (((\mathbf{u}_m - \mathbf{u}^*) \cdot \nabla)\mathbf{u}_m, \mathbf{v}) + ((\mathbf{u}^* \cdot \nabla)(\mathbf{u}_m - \mathbf{u}^*), \mathbf{v})$. By (12), using Lemma 1.1 and (11), we obtain that

$$|(((\mathbf{u}_m - \mathbf{u}^*) \cdot \nabla)\mathbf{u}_m, \mathbf{v})| \leq \gamma'_1 \|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3} \|\mathbf{u}_m\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Using (5), we deduce that

$$|((\mathbf{u}^* \cdot \nabla)\mathbf{v}, \mathbf{u}_m - \mathbf{u}^*)| \leq \gamma'_1 \|\mathbf{u}^*\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

For a nonlinear term $(\mathbf{u}_m \cdot \nabla\varphi_m, h)$ in (14), the equality holds

$$(\mathbf{u}_m \cdot \nabla\varphi_m, h) - (\mathbf{u}^* \cdot \nabla\varphi^*, h) = ((\mathbf{u}_m - \mathbf{u}^*) \cdot \nabla\varphi_m, h) + (\mathbf{u}^* \cdot \nabla(\varphi_m - \varphi^*), h).$$

From (12), using Lemma 2.1 and (11), we arrive at

$$|((\mathbf{u}_m - \mathbf{u}^*) \cdot \nabla\varphi_m, h)| \leq \gamma'_2 \|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3} \|\varphi_m\|_{1,\Omega} \|h\|_{1,\Omega} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

As $\mathbf{u}^* h \in L^3(\Omega)^3$, then from (12) it follows that

$$(\mathbf{u}^* \cdot \nabla(\varphi_m - \varphi^*), h) = (\nabla(\varphi_m - \varphi^*), h \mathbf{u}^*) \rightarrow 0 \text{ as } m \rightarrow \infty \forall h \in \mathcal{T}.$$

For a nonlinear term $(\lambda\alpha_m\varphi_m, h)_{\Gamma_N}$ we have that

$$(\lambda\alpha_m\varphi_m, h)_{\Gamma_N} - (\lambda\alpha^*\varphi^*, h)_{\Gamma_N} = (\lambda\alpha_m(\varphi_m - \varphi^*), h)_{\Gamma_N} + (\alpha_m - \alpha, \lambda\varphi^*h)_{\Gamma_N}. \tag{15}$$

As $\lambda\varphi^*h \in L^2(\Gamma_N)$, then for the second term in right-hand side of (15) we have

$$(\alpha_m - \alpha^*, \lambda^*\varphi^*h)_{\Gamma_N} \rightarrow 0 \text{ as } m \rightarrow \infty \forall h \in \mathcal{T}.$$

Since $C^\infty(\bar{\Omega})$ is dense in \mathcal{T} , there exists a sequence $h_n \in C^\infty(\bar{\Omega})$, converging to h as $n \rightarrow \infty$ by the norm $\|\cdot\|_{1,\Omega}$. With the help of $\{h_n\}$ we have

$$(\lambda\alpha_m(\varphi_m - \varphi^*), h)_{\Gamma_N} = (\lambda\alpha_m(\varphi_m - \varphi^*), h_n)_{\Gamma_N} + (\lambda\alpha_m(\varphi_m - \varphi^*), h - h_n)_{\Gamma_N} \quad \forall m, n \in \mathbf{N}. \tag{16}$$

By the uniform boundedness over m of the quantities $\|\alpha_m\|_{\Gamma_N}$ and $\|\varphi_m - \varphi^*\|_{L^4(\Omega)}$, there exists a number $N = N(\varepsilon, h)$, such that the second term in (16) satisfies as $n \geq N, m \in \mathbf{N}$

$$|(\lambda^*\alpha_m(\varphi_m - \varphi^*), h - h_n)_{\Gamma_N}| \leq \|\lambda^*\|_{L^\infty(\Omega)} \|\alpha_m\|_{\Gamma_N} \|\varphi_m - \varphi^*\|_{L^4(\Omega)} \|h_n - h\|_{L^4(\Omega)} \leq \varepsilon/2. \tag{17}$$

By the uniform boundedness over m of the quantity $\|\alpha_m\|_{L^2(\Gamma_N)}$ and convergence of $\|\varphi_m - \varphi^*\|_{L^3(\Gamma_N)}$ there exists a number $M = M(\varepsilon, h)$, such that the first term in (17) satisfies

$$|(\lambda\alpha_m(\varphi_m - \varphi^*), h_n)_{\Gamma_N}| \leq \|\lambda^*\|_{L^\infty(\Omega)} \|\alpha_m\|_{\Gamma_N} \|\varphi_m - \varphi^*\|_{L^3(\Omega)} \|h_n\|_{L^6(\Omega)} \leq \varepsilon/2, \quad m \geq M, n \in \mathbf{N}. \tag{18}$$

Then from (17) and (18) it follows that $|(\lambda^*\alpha_m(\varphi_m - \varphi^*), h)_{\Gamma_N}| \rightarrow 0$ as $m \rightarrow \infty \forall h \in \mathcal{T}$.

As the functional J is weakly semicontinuous below on $X \times L^2(\Gamma_N)$, then from (11) it follows that $J(\mathbf{x}^*, \alpha^*) = J^*$. ■

Remark 2.1. It is clear, that all cost functionals from (10) satisfy the conditions of the Theorem 2.1.

In subsequent papers, optimality systems will be derived for the control problem (9). Based on their analysis questions of uniqueness and stability of optimal solutions will be studied and numerical algorithms for solving control problems will be constructed.

Acknowledgments

The first author was supported by the state assignment of Institute of Applied Mathematics FEB RAS (Theme No. 075-01095-20-00), the second author was supported by the Ministry of Science and Higher Education of the Russian Federation (project no. 075-02-2020-1482-1, additional agreement of 21.04.2020).

References

- [1] Ito K and Kunish K 1997 *Inverse Probl.* **14** 995–1013
- [2] Agoshkov V I, Minuk F P, Rusakov A S and Zalesny V B 2005 *Russ. J. Numer. Anal. Math. Model.* **20** 19–43
- [3] Nguyen P A and Raymond J -P 2001 *ESAIM Contr. Op. Ca. Va.* **6** 467–88
- [4] Alekseev G V and Adomavichus D A 2001 *J. Inverse Ill-Posed Probl.* **9** 435–68
- [5] Alekseev G V 2001 *Sib. Math. J.* **42** 811–27
- [6] Alekseev G V 2002 *Comp. Math. Math. Phys.* **42** 363–76
- [7] Alekseev G V, Soboleva O V and Tereshko D A 2008 *J. Appl. Mech. Tech. Phys.* **49** 537–47
- [8] Nguyen P A and Raymond J -P 2011 *Syst. Control Lett.* **60** 249–55
- [9] Brizitskii R V, Saritskaya Zh Yu and Byrganov A I 2016 *Sib. Electron. Math. Rep.* **13** 352–60
- [10] Alekseev G V, Brizitskii R V and Saritskaya Zh Yu 2016 *J. Appl. Indust. Math.* **10** 155–67
- [11] Brizitskii R V and Saritskaya Zh Yu 2016 *Comp. Math. Math. Phys.* **56** 2011–22
- [12] Brizitskii R V and Saritskaya Zh Yu 2017 *Diff. Eq.* **53** 485–96
- [13] Brizitskii R V and Saritskaya Zh Yu 2018 *Izv. Math.* **82** 14–30
- [14] Brizitskii R V, Saritskaya Zh Yu and Kravchuk R R 2019 *Sib. Electron. Math. Rep.* **16** 1215–32
- [15] Brizitskii R V and Saritskaya Zh Yu 2018 *J. Inverse Ill-Posed Probl.* **9** 821–34
- [16] Alekseev G V and Levin V A 2016 *Dokl. Phys.* **61** (11) 546–50
- [17] Alekseev G V 2014 *Comp. Math. Math. Phys.* **54** 1788–803
- [18] Alekseev G V 2016 *Diff. Eq.* **52** 361–72
- [19] Soboleva O V, Brizitskaya A V and Dyachenko Yu K 2020 *IOP Conf. Ser.: Earth Environ. Sci.* **459** 042062
- [20] Lorca S A and Boldrini J L 1996 *J. Diff. Eq.* **124** 389–406
- [21] Bermudez A, Munoz-Sola R and Vazquez R 2010 *J. Math. Anal. Appl.* **368** 444–68
- [22] Růžička M, Shelukhin V and dos Santos M M 2017 *Math. Meth. Appl. Sc.* **40** 2746–61
- [23] Mamontov A E and Prokudin D A 2018 *Izv. Math.* **82** 140–85
- [24] Kovtanyuk A E, Chebotarev A Yu, Botkin N D and Hoffman K-H 2014 *Appl. Math. Comp.* **249** 247–52
- [25] Grenkin G V, Chebotarev A Y, Kovtanyuk A E, Botkin N D and Hoffmann K-H 2016 *J. Math. Anal. Appl.* **433** 1243–60